Superradiance of a subwavelength array of classical nonlinear emitters

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Abstract: We suggest a mechanism by which a superradiant burst emerges from a subwavelength array of nonlinear classical emitters that are not initially synchronized. The emitters interact via the field of their common radiative response. We show that only if the distribution of initial phases is not uniform does a non-zero field of radiative response arise, leading to a superradiant burst. Although this field cannot synchronize the emitters, it engenders long period envelopes for their fast oscillations. Constructive interference in the envelopes of several emitters creates a large fluctuation in dipole moments that results in a superradiant pulse. The intensity of this pulse is proportional to the square of the number of emitters participating in the fluctuation.

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References and links

1. L. Allen and J. Eberly, *Optical Resonance and Two-Level Atoms* (Dover Publications, 1987).
2. A. V. Andreev, V. I. Emel'yanov, and Y. A. Il'inskiĭ, "Collective spontaneous emission (Dicke superradiance)," Sov. Phys. Usp. **23**(8), 493–514 (1980).
3. M. Gross and S. Haroche, "Superradiance: An essay on the theory of collective spontaneous emission," Phys. Rep. **93**(5), 301–396 (1982).
4. J. D. Macomber, *The Dynamics of Spectroscopic Transitions* (Wiley, 1976).
5. L. I. Men'shikov, "Superradiance and related phenomena," Phys. Uspekhi **42**(2), 107–147 (1999).
6. P. P. Vasil'ev, "Femtosecond superradiant emission in inorganic semiconductors," Rep. Prog. Phys. **72**(7), 076501 (2009).
7. P. P. Vasil'ev, R. V. Penty, and I. H. White, "Superradiant emission in semiconductor diode laser structures," IEEE J. Sel. Top. Quantum Electron. **19**(4), 1500210 (2013).
8. R. H. Dicke, "Coherence in spontaneous radiation processes," Phys. Rev. **93**(1), 99–110 (1954).
9. I. E. Protsenko and A. V. Uskov, "Superradiance of several atoms near a metal nanosphere," Quant. Electron. **45**(6), 561–572 (2015).
10. V. N. Pustovit and T. V. Shahbazyan, "Cooperative emission of light by an ensemble of dipoles near a metal nanoparticle: the plasmonic Dicke effect," Phys. Rev. Lett. **102**(7), 077401 (2009).
11. M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, "Observation of Bose-Einstein condensation in a dilute atomic vapor," Science **269**(5221), 198–201 (1995).
12. C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, "Evidence of Bose-Einstein condensation in an atomic gas with attractive interactions," Phys. Rev. Lett. **75**(9), 1687–1690 (1995).
13. S. Inouye, A. P. Chikkatur, D. M. Stamper-Kurn, J. Stenger, D. E. Pritchard, and W. Ketterle, "Superradiant rayleigh scattering from a Bose-Einstein condensate," Science **285**(5427), 571–574 (1999).
14. N. Bar-Gill, E. E. Rowen, and N. Davidson, "Spectroscopy of strong-pulse superradiance in a Bose-Einstein condensate," Phys. Rev. A **76**(4), 043603 (2007).
15. A. Hilliard, F. Kaminski, R. le Targat, C. Olausson, E. S. Polzik, and J. H. Müller, "Rayleigh superradiance and dynamic Bragg gratings in an end-pumped Bose-Einstein condensate," Phys. Rev. A **78**(5), 051403 (2008).
16. G. O. Ariunbold, W. Yang, A. V. Sokolov, V. A. Sautenkov, and M. O. Scully, "Picosecond superradiance in a three-photon resonant medium," Phys. Rev. A **85**(2), 023424 (2012).

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17. J. G. Bohnet, Z. Chen, J. M. Weiner, D. Meiser, M. J. Holland, and J. K. Thompson, “A steady-state superradiant laser with less than one intracavity photon,” Nature **484**(7392), 78–81 (2012).
18. S. Bux, H. Tomczyk, D. Schmidt, P. W. Courteille, N. Piovella, and C. Zimmermann, “Control of matter-wave superradiance with a high-finesse ring cavity,” Phys. Rev. A **87**(2), 023607 (2013).
19. H. Keßler, J. Klinder, M. Wolke, and A. Hemmerich, “Steering matter wave superradiance with an ultranarrow-band optical cavity,” Phys. Rev. Lett. **113**(7), 070404 (2014).
20. J. Klinder, H. Keßler, M. Wolke, L. Mathey, and A. Hemmerich, “Dynamical phase transition in the open Dicke model,” Proc. Natl. Acad. Sci. U.S.A. **112**(11), 3290–3295 (2015).
21. P. P. Vasil’ev, V. Olle, R. V. Penty, and I. H. White, “Experimental study of delay-time statistics of superradiant pulses generated from semiconductor structures,” Appl. Phys. Lett. **103**(24), 241108 (2013).
22. J. A. Mlynek, A. A. Abdumalikov, C. Eichler, and A. Wallraff, “Observation of Dicke superradiance for two artificial atoms in a cavity with high decay rate,” Nat. Commun. **5**, 5186 (2014).
23. S. D. Boer and D. A. Wiersma, “Dephasing-induced damping of superradiant emission in j-aggregates,” Chem. Phys. Lett. **165**(1), 45–53 (1990).
24. F. Meinardi, M. Cerminara, A. Sassella, R. Bonifacio, and R. Tubino, “Superradiance in Molecular H Aggregates,” Phys. Rev. Lett. **91**(24), 247401 (2003).
25. N. E. Rehler and J. H. Eberly, “Superradiance,” Phys. Rev. A **3**(5), 1735–1751 (1971).
26. E. Ressayre and A. Tallet, “Quantum theory for superradiance,” Phys. Rev. A **15**(6), 2410–2423 (1977).
27. M. G. Benedict, A. M. Ermolaev, V. A. Malyshev, I. V. Sokolov, and E. D. Trifonov, *Super-radiance: Multiatomic Coherent Emission* (IOP, 1996).
28. R. Bonifacio, P. Schwendimann, and F. Haake, “Quantum statistical theory of superradiance. I,” Phys. Rev. A **4**(1), 302–313 (1971).
29. R. J. Glauber and F. Haake, “Superradiant pulses and directed angular momentum states,” Phys. Rev. A **13**(1), 357–366 (1976).
30. R. Bonifacio, P. Schwendimann, and F. Haake, “Quantum statistical theory of superradiance. II,” Phys. Rev. A **4**(3), 854–864 (1971).
31. M. G. Moore and P. Meystre, “Theory of superradiant scattering of laser light from Bose-Einstein condensates,” Phys. Rev. Lett. **83**(25), 5202–5205 (1999).
32. J. Keeling, M. J. Bhaseen, and B. D. Simons, “Fermionic superradiance in a transversely pumped optical cavity,” Phys. Rev. Lett. **112**(14), 143002 (2014).
33. Y. Chen, Z. Yu, and H. Zhai, “Superradiance of degenerate Fermi gases in a cavity,” Phys. Rev. Lett. **112**(14), 143004 (2014).
34. T. Brandes, “Coherent and collective quantum optical effects in mesoscopic systems,” Phys. Rep. **408**(5-6), 315–474 (2005).
35. S. Ashhab, “Superradiance transition in a system with a single qubit and a single oscillator,” Phys. Rev. A **87**(1), 013826 (2013).
36. D. Bates and B. Bederson, eds., *Atomic and Molecular Physics* (Academic Press, 1981).
37. A. B. Bhattacherjee, “Dicke–Hepp–Lieb superradiant phase transition and independent modes model in quantum optomechanics,” Optik (Stuttg.) **124**(21), 5267–5270 (2013).
38. K. Hepp and E. H. Lieb, “On the superradiant phase transition for molecules in a quantized radiation field: the Dicke maser model,” Ann. Phys. **76**(2), 360–404 (1973).
39. T. Liu, Y.-Y. Zhang, Q.-H. Chen, and K.-L. Wang, “Possibility of superradiant phase transitions in coupled two-level atoms,” Phys. Lett. A **376**(26-27), 1962–1965 (2012).
40. M. O. Scully, E. S. Fry, C. H. R. Ooi, and K. Wódkiewicz, “Directed spontaneous emission from an extended ensemble of N atoms: timing is everything,” Phys. Rev. Lett. **96**(1), 010501 (2006).
41. A. A. Svidzinsky, J.-T. Chang, and M. O. Scully, “Dynamical evolution of correlated spontaneous emission of a single photon from a uniformly excited cloud of N atoms,” Phys. Rev. Lett. **100**(16), 160504 (2008).
42. M. O. Scully and A. A. Svidzinsky, “The effects of the N atom collective Lamb shift on single photon superradiance,” Phys. Lett. A **373**(14), 1283–1286 (2009).
43. R. Friedberg and J. T. Manassah, “Effects of including the counterrotating term and virtual photons on the eigenfunctions and eigenvalues of a scalar photon collective emission theory,” Phys. Lett. A **372**(14), 2514–2521 (2008).
44. A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization. A Universal Concept in Nonlinear Sciences* (Cambridge University Press, 2003).
45. A. V. Dorofeenko, A. A. Zyablovsky, A. P. Vinogradov, E. S. Andrianov, A. A. Pukhov, and A. A. Lisyansky, “Steady state superradiance of a 2D-spaser array,” Opt. Express **21**(12), 14539–14547 (2013).
46. T. Pickering, J. M. Hamm, A. F. Page, S. Wuestner, and O. Hess, “Cavity-free plasmonic nanolasing enabled by dispersionless stopped light,” Nat. Commun. **5**, 4972 (2014).
47. C. A. Balanis, *Antenna Theory* (Wiley, 2005).
48. W. L. Stutzman and G. A. Thiele, *Antenna Theory and Design* (Wiley, 2012).
49. W. Zakowicz and K. Rzazewski, “Collective radiation by harmonic oscillators,” J. Phys. A **7**(7), 869–880 (1974).
50. A. A. Svidzinsky, J.-T. Chang, and M. O. Scully, “Cooperative spontaneous emission of N atoms: Many-body eigenstates, the effect of virtual Lamb shift processes, and analogy with radiation of N classical oscillators,” Phys. Rev. A **81**(5), 053821 (2010).

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51. L. A. Vainstein and A. I. Kleev, "Cooperative radiation of electron-oscillators," Sov. Phys. Dokl. **35**(4), 359–361 (1990).
52. I. Tralle and P. Zięba, "Induced -cooperative phenomenon in an ensemble of the nonlinear coupled oscillators," Phys. Lett. A **378**(20), 1364–1368 (2014).
53. Y. A. Il'inskii and N. S. Maslova, "Classical analog of superradiance in a system of interacting nonlinear oscillators," Sov. Phys. JETP **67**(1), 96–97 (1988).
54. A. V. Dorofeenko, A. A. Zyablovsky, A. A. Pukhov, A. A. Lisyansky, and A. P. Vinogradov, "Light propagation in composite materials with gain layers," Phys. Uspekhi **55**(11), 1080–1097 (2012).
55. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Butterworth-Heinemann, 1980).
56. J. A. Stratton, *Electromagnetic Theory* (Wiley-IEEE Press, 2007).
57. A. H. Nayfeh, and D. T. Mook, *Nonlinear Oscillations* (Wiley, 1979).

1. Introduction

Superradiance (SR) is a cooperative spontaneous radiation of photons by an array of emitters coupling via a common light field. Different aspects of this phenomenon are reviewed in Refs [1–7]. For a subwavelength array of quantum emitters, SR was predicted by Dicke [8]. The Dicke model assumes that identical quantum emitters interact via the common field of their radiative response. Once excited, the emitters form a special Dicke state [3] and after some delay radiate a pulse. It is commonly considered that this delay is a phenomenon that arises as a result of the quantum description of dynamics of the Dicke state. Immediately on excitation, the dipole moment of a system of N identical two-level atoms is zero [1,4]. During the radiation process, the dipole moment amplitude increases. When it reaches its maximum value, the superradiant burst appears. The duration of the SR burst is smaller than the radiation time of a single emitter by a factor of $1/N$. The intensity of the radiation is proportional to N^2 and the delay time $t_0 \sim \log N / N$. The Dicke approach only considers the dynamics of the population inversion of quantum emitters without discussing their phases. The intensity of SR radiation is defined as the time derivative of the total inversion. This is correct only if all emitters have the same phase. The formal Dicke approach does not allow one to elucidate the physical mechanism of the emitter synchronization.

Recently [9,10], an unusual realization of the Dicke SR has been theoretically demonstrated for a pulse-excited complex spaser, in which an ensemble of quantum emitters is placed near a metal nanoparticle. In this case, the plasmon excitation takes place instead of radiation of photons. The SR manifests itself in a quick, comparing to the usual rate of plasmon excitation, burst of plasmons. For the phenomenon to occur, all emitters must initially be in the excited state. In the other words, to exhibit SR, the system must be in the Dicke state

Experimentally creating a Dicke state is not easy. There are only few quantum systems in which one can correlate all atoms in the initial moment. One of them is the Bose-Einstein condensate. It can be created by cooling atoms in a quadruple trap [11,12]. The condensate is described by a common wave function, and even though it differs from a system of two-level atoms, a phenomenon similar to SR has been observed [13–20]. Important systems in which SR has been observed experimentally are semiconductor structures. In Refs [6,7,21]. it was shown that in GaAs/AlGaAs heterostructures, the exciton condensate of e-h pairs can be used to form the Dicke state. If the carrier density reaches the critical value, the condensed e-h state may be formed in the time interval sufficient for an SR burst. It has also been shown that such an SR generation is possible for room temperature.

In these experiments, however, SR has a more complicated multiple peak structure in contrast to the single peak predicted by Dicke. A correlated quantum state can also be realized in a system of superconducting qubits. It was shown that in a system of two superconducting qubits an intensity increase and a delay time of the pulse occur [22].

A phenomenon similar to SR has been demonstrated in a system of organic molecules, *J*-aggregates [23] and *H*-aggregates [24]. In these systems, initially correlated states do not

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seem exist. Nevertheless, a delayed radiation peak has been observed. This is not a single peak, however, and its intensity is not proportional to N^2.

In large-scale systems with sizes greatly exceeding the optical wavelength, at least in one dimension, phenomena similar to SR have also been observed experimentally [3,25,26]. The results of these experiments are not unambiguous because, in such systems, a significant contribution to radiation may arise from waves originated at one end and then radiated at the other [3,27]. Thus, it is hard to separate the contributions of SR and stimulated emission [3,25,26].

At the same time, a number of theoretical considerations have focused on increasing the accuracy of the description of a quantum system [3,28,29]. For example, an approach based on the density matrix formalism [3,28,30] makes obtained results more rigorous but does not reveal their physics. In any case, it has been commonly recognized that for SR to occur the quantum emitters need to be in the Dicke state. Much attention has also been paid to the mechanism of emitter synchronization or the formation of the Dicke state. This can be achieved by forming a Bose-Einstein condensate of emitters [31–34]. Another actively explored way for achieving the Dicke state is by a so-called Dicke–Hepp–Lieb superradiant phase transition of a two-level system in an external electromagnetic field at finite temperature [35–39]. Interesting theoretical studies have been conducted on the excitation by a single photon of *N* being in the the Dicke state [40,41]. The radiation rate of such a system increases by the factor of *N* with no delay time [42,43].

A system in which synchronization of emitters can be achieved straightforwardly is a nonlinear auto-oscillating system with continuous excitation. Emitters in such a system can be synchronized by an external driving force. In this case, all auto-oscillations have the phase and frequency of this force [44–46]. In Refs [44–46], the field of the response radiation of a collection of auto-oscillating systems (spasers) has been considered as a synchronizing factor. The radiation intensity has been predicted to be proportional to the square of the number of emitters. However, since the synchronized auto-oscillating process is stationary, it cannot give rise to an SR pulse.

An SR phenomenon is also known for classical systems. According to the antenna theory [47, 48], a subwavelength system of *N* classical emitters oscillating in phase loses energy *N* times faster than a single oscillator. Therefore, when emitters are in the "classical" Dicke state, the intensity of the radiation peak increases by a factor of N^2 similar to a quantum system [49–51]. SR in a system of classical nonlinear oscillators synchronized by an external force has been demonstrated numerically [52].

An extensive study of SR in classical systems was carried out by Vainshtein and Kleev [51]. In this study, similar to quantum emitters, an ensemble of classical nonlinear emitters interacting via their common radiation field was shown to radiate an SR pulse [51,53]. The Vainshtein-Kleev model assumes that initially excited classical emitters, which are not pumped further, have a random distribution of phases. Computer simulation [51] showed that classical linear emitters do not become superradiant, while in the case of cubic nonlinearity one pulse or a sequence of pulses may arise depending on the initial phase distribution realization. The linear analysis [53] shows that an initial state with randomly distributed phases of emitters is unstable. In Ref [53]. the instability was interpreted as emitter phasing. This is similar to quantum oscillators, for which the response to a self-consistent field is always nonlinear [2,5,54]. Again, the main question is how excited emitters initially having different phases evolve into the Dicke state and produce in-phase oscillations.

In this paper, we demonstrate that an initial Dicke state is not necessary for SR in classical systems and suggest a mechanism for the emergence of an SR pulse in an ensemble of classical nonlinear dipole emitters. Following Dicke [8] and Vainshtein-Kleev [51], we assume that each dipole is in the total field of the radiative response of the whole system. This field is produced by all dipoles and depends on their phase distribution. We show that this field may arise only due to a fluctuation in the dipole phase distribution that is initially

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uniform. This field causes a modulation of the fast oscillations of dipoles with a periodic envelope. The frequency of the envelope is determined by the initial phase of the dipole oscillation. This frequency is much smaller than the frequency of the dipole oscillation. SR arises due to constructive interference in long-period envelopes of fast oscillations which causes an increase in the amplitude of the oscillation of the total dipole moment of the system. The lifetime of the large amplitude oscillations is of the order of the envelope period which is much greater than the period of fast oscillations.

2. Dynamics of interacting classical nonlinear dipoles

To begin, we briefly discuss the results obtained in Ref [51]. We consider a system of oscillating dipoles placed in a subwavelength volume $V << \lambda^3$. We assume that the energy of dipoles oscillating with the frequency ω is much greater than $\hbar\omega$, so that the classical theory is applicable. We use the model suggested in Refs [51, 53]. to describe the dynamics of these oscillators.

The field of the each oscillator can be expressed via the Hertz vector [55,56]:

$$\mathbf{\Pi} = -\frac{1}{r}\mathbf{d}\left(t - \frac{r}{c}\right), \tag{1}$$

where **d** is the dipole moment of an oscillator and **r** is the distance from the oscillator to the observation point. The Fourier component of the vector $\mathbf{\Pi}$ has the form

$$\Pi_\omega = -\frac{\mathbf{d}_\omega \exp(ik_0 r)}{r}, \tag{2}$$

where $k_0 = \omega / c$. Since we are interested in the field at small distances, $k_0 r << 1$, we expand $\mathbf{\Pi}_\omega$ into a series

$$\mathbf{\Pi}_\omega = -\mathbf{d}_\omega\left(1 + ik_0 r - k_0^2 r^2 / 2 - ik_0^3 r^3 / 6 + ...\right) / r. \tag{3}$$

Assuming that all dipoles are directed along the *z*-axis we find the *z*-component of the electric field:

$$E_\omega^z = -k^2 \Pi_\omega^z - \Delta\Pi_\omega^z = d_\omega\left(\frac{3z^2 - r^2}{r^5} + \frac{k_0^2(r^2 + z^2)}{2r^3} + \frac{2ik_0^3}{3} + ...\right). \tag{4}$$

Since $d = ez$ is the dipole moment of a charge oscillating along the *z*-axis and $i^n \partial^n z / c^n \partial t^n$ can substitute for zk_0^n, we obtain

$$E_z = e\left[\frac{1 - 3\cos^2\alpha}{cr^2} - \frac{1 + \cos^2\alpha}{2c^2 r}\frac{\partial^2 z(t)}{\partial t^2} + \frac{2}{3c^3}\frac{\partial^3 z(t)}{\partial t^3} + \ldots\right], \tag{5}$$

where $z(t)$ is an instantaneous position of the oscillator (dipole) and α is the angle between **r** and the *z*-axis [55,56]. Since the oscillators are confined in a subwavelength volume, the retardation effects can be neglected. We omit, therefore, the terms with derivatives higher than $\partial^3 z / \partial t^3$ in Eq. (5). In this equation, the first term corresponds to the quasistatic Coulomb field and the second term, proportional to $1/r$, describes the induction field. These fields suppress SR [5]. Since our goal is revealing the SR mechanism, we eliminate all effects masking SR. For this purpose, we consider and idealized system in which emitters are positioned either in a circle [3] or form an ideal cubic lattice. Of course, it is difficult to realize a system of either of these symmetries in experiment. However, for these symmetries, the fields associated with the first and second terms in the right-hand side of Eq. (5) turn to

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zero, eliminating effects that affect SR. Considering systems of more realistic symmetry makes the effect weaker and calculations much more cumbersome. The third term,

$$E_z(t) = \frac{2e}{3c^3}\frac{\partial^3 z(t)}{\partial t^3}, \quad (6)$$

expresses the field of the radiative response. Due to its active nature, it is in an anti-phase with the dipole current. SR cannot be observed for a system of linear dipoles [51]. The simplest way to take nonlinearity into account is by adding a cubic term into the equation of motion. A physical reason for a dipole nonlinear response on an external perturbation could be either a finite number of excitation levels or electron radiation due to its motion in a magnetic field. The equation of motion of the nonlinear oscillator due to field (6) has the form

$$\frac{\partial^2 z}{\partial t^2} + \omega^2 z + \mu z_k^3 = t_e \frac{\partial^3 z}{\partial t^3}, \quad (7)$$

where $t_e = (2e^2)/(3mc^3) = (2r_0)/3c = 6.27\cdot 10^{-24} s$, and $r_0 = e^2/mc^2 = 2.82\cdot 10^{-13} cm$ is the classical radius of an electron. Using the smallness of t_e to estimate $\partial^3 z/\partial t^3$ we can reduce the right-hand side in Eq. (7) to $\partial^3 z/\partial t^3 \approx -\omega^2 \partial z/\partial t$. This can be done when $t_e << \omega^{-1}$. Since $t_e \sim 10^{-23} s$, this condition is fulfilled for both radio and optical dipole frequencies. We assume that the parameter of nonlinearity, μ, is small, so that $\omega^2 >> \mu z^2$, where z is a characteristic dipole amplitude (e.g., its initial value). Then, Eq. (7) takes the form

$$\frac{\partial^2 z}{\partial t^2} + \omega^2 z + \mu z_k^3 = -t_e \omega^2 \frac{\partial z}{\partial t}. \quad (8)$$

For one dipole, the field of the radiative response is defined by Eq. (6). For $\omega r/c >> 1$, field (6) does not depend on a distance and for N dipoles we have

$$E_z = \left(2e/3c^3\right) N \left\langle \frac{\partial^3 z}{\partial t^3} \right\rangle, \quad z = N^{-1}\sum_{n=1}^{N} z_n.$$

Then for an ensemble of N nonlinear dipoles we obtain a system of equations

$$\frac{\partial^2 z_k}{\partial t^2} + \omega^2 z_k + \mu z_k^3 = -N\nu \left\langle \frac{\partial z}{\partial t} \right\rangle, \quad (9)$$

where $\nu = 2e^2\omega^2/3mc^3$.

Now, instead of real quantities z_k we introduce complex dimensionless variables, envelopes c_k:

$$z_k(t) = a\left(c_k(t)e^{-i\omega_0 t} + c_k^*(t)e^{i\omega_0 t}\right)/2 = a\mathrm{Re}\left(c_k(t)e^{-i\omega_0 t}\right) \quad (10)$$

In Eq. (11), a is the initial amplitude of oscillations, $a = |z_k(0)|$, which is the same for all k. As mentioned above, this amplitude is constrained by the inequality $\mu a^2 << \omega^2$. Now the dynamics of each dipole is defined by the quantity $c_k(t) = |c_k(t)|\exp\left(i\varphi_k(t)\right)$, where $\varphi_k(t) = \arg\left(c_k(t)\right)$.

Due to the nonlinear character of Eq. (10), by using Eq. (11) we introduce two variables $|c_k(t)|$ and $\varphi_k(t)$ (or $\mathrm{Re}\, c_k$ and $\mathrm{Im}\, c_k$) instead of one real variable $z_k(t)$. By solving Eq. (10) one cannot find both functions. It is only possible for a linear equation with constant

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coefficients when $\varphi_k(t) = \text{const}$. To eliminate the arisen ambiguity, we should impose an additional condition. A convenient choice of such a condition is (for details see Refs [57].):

$$\frac{\partial c_k}{\partial t} e^{-i\omega_0 t} + \frac{\partial c_k^*}{\partial t} e^{i\omega_0 t} = 0. \tag{11}$$

This additional condition gives us the second equation to determine both unknown functions $|c_k(t)|$ and $\varphi_k(t)$. Moreover, the choice of this condition in form (12) allows one to eliminate terms containing second derivatives from Eq. (10) after substitution (11) is performed. Indeed, differentiating Eq. (11) we obtain

$$\frac{\partial z_k(t)}{\partial t} = \frac{-i\omega_0 a}{2}\left(c_k(t)e^{-i\omega_0 t} - c_k^*(t)e^{i\omega_0 t}\right) + \frac{a}{2}\left(\frac{\partial c_k(t)}{\partial t} e^{-i\omega_0 t} + \frac{\partial c_k^*(t)}{\partial t} e^{i\omega_0 t}\right).$$

Thanks to condition (12), the last term vanishes from this equation. Therefore, $\partial z_k(t)/\partial t$ does not have terms proportional to $\partial c_k(t)/\partial t$ and $\partial^2 z_k(t)/\partial t^2$ does not contain terms proportional to $\partial^2 c_k(t)/\partial t^2$. By using Eqs. (11) and (12) one can reduce Eq. (10) to

$$-i\omega_0 \dot{c}_k + \frac{1}{2}(\omega^2 - \omega_0^2)c_k + \frac{1}{2}(\omega^2 - \omega_0^2)c_k^* e^{2i\omega_0 t} + \frac{\mu a^2}{8}(c_k^3 e^{-2i\omega_0 t} + 3c_k^2 c_k^* + 3c_k c_k^{*2} e^{2i\omega_0 t} + 3c_k^{*3} e^{4i\omega_0 t})$$
$$= -\frac{\nu}{2}\sum_k (-i\omega_0 c_k + i\omega_0 c_k^* e^{2i\omega_0 t}).$$

Then, by averaging this equation over the period of fast oscillations, $T = 2\pi/\omega_0$, and assuming $\omega_0^2 = \omega^2 - 3\mu^2 a^2/8$ we obtain one equations for a complex unknown or two equations for two real unknowns:

$$\frac{\partial c_k}{\partial t} + i\chi\omega_0\left(|c_k|^2 - 1\right)c_k = -\frac{1}{2}N\nu\langle c\rangle, \tag{12}$$

where $\chi = 3\mu a^2/8\omega_0^2$ is the coefficient of non-isochronism [53]. By using dimensionless parameters,

$$\tau = t/\tau_N = N\nu t/2, \quad \tau_N = 2/(N\nu), \quad \theta = 2\chi\omega_0/(N\nu), \tag{13}$$

for slowly varying amplitudes, Eqs. (13) can be simplified

$$\frac{dc_k}{d\tau} + i\theta\left(|c_k|^2 - 1\right)c_k = -\langle c\rangle, \tag{14}$$

where the parameter θ may be either positive or negative. Equation (15) with $\theta = 0$ describes linear oscillators. Since we assume that in the initial moment of time $|c_k(0)| = 1$ for all emitters, their dynamics is defined by initial phases, $c_k(0) \sim \exp(i\varphi_k(0))$. So far, the frequency ω_0 is arbitrary; its value is determined later from convenience considerations. Note that all phase detunings are included into the complex amplitude. Fast oscillations that fill the envelope are in phase.

When choosing ω_0 as $\omega_0^2 = \omega^2 - 3\mu^2 a^2/8$, in the initial moment of time, in which $|c_k(0)|^2 = 1$, Eq. (15) becomes linear. This allows us to develop a mean field theory (see Sec. 3).

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In Ref [51], for computer simulations of Eq. (15), the initial phase of each emitter has randomly been chosen from the interval $[0, 2\pi]$ employing uniform probability distribution. The obtained results qualitatively agree with the Dicke model. This solution shows one or more delayed SR pulses. The number of pulses, their intensity, and the delay times depend on a particular realization of initial dipole phases. To elucidate the mechanism of SR and the random character of peaks, we conduct additional studies. The results of our computer simulations for dependencies of the average intensity, $\left|\langle c_k \rangle\right|^2$, and the average energy, $\left\langle |c_k|^2 \right\rangle$, on time qualitatively agree with that of Ref [51]. and are shown in Fig. 1.

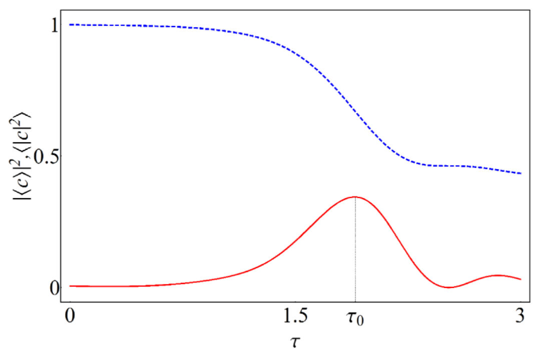

Fig. 1. The average intensity, $\left|\langle c_k \rangle\right|^2$, (the solid red line) and the average energy, $\left\langle |c_k|^2 \right\rangle$, (the dashed blue line) as functions of time τ calculated for the random distribution of initial phases. The value of $\theta = 10$ and the number of dipoles $N = 250$ are the same as in Ref [51]. The delay time of the superradiant burst is $\tau_0 \approx 1.9$.

The dependencies shown in Fig. 1 are in qualitative agreement with the usual picture of the Dicke SR in which the first large peak of the intensity has the duration of $\tau_s \tau_N \sim 1/N$, and its delay time is $\tau_0 \sim \log N$ (see Fig. 2). Note that the dimensional delay time has the same dependency on the number of emitters as in the Dicke model, $t_0 = \tau_N \tau_0 \sim \log N / N$.

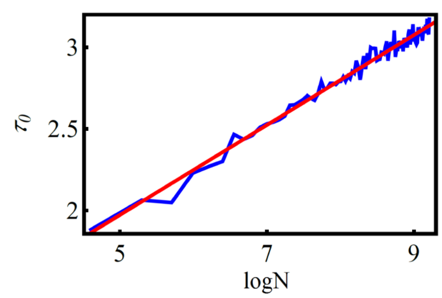

Fig. 2. The dependence of the delay time $\tau_0 = t_0 / \tau_N$ on the number of dipoles for the random initial phase distribution in the interval $[-\pi, \pi]$. The approximation line $\alpha_{\tau_0} \log N$ with $\alpha_{\tau_0} \approx 0.5$ is shown in red.

Even though our computer simulation predicts an existence of an SR pulse and a delay time, it differs from the intensity dependencies on N predicted by Dicke. In the Dicke model, the intensity of the SR peak is proportional to N^2, while our numerical calculations do not

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show a power dependence of the intensity on the number of emitters (see Fig. 3). Moreover, we obtain that a realization of the SR burst depends on a particular phase distribution at the initial time. In contrast to the Dicke model, we can observe one burst or a set of bursts. Furthermore, for the regular uniform distribution of initial phases, the burst does not arise at all.

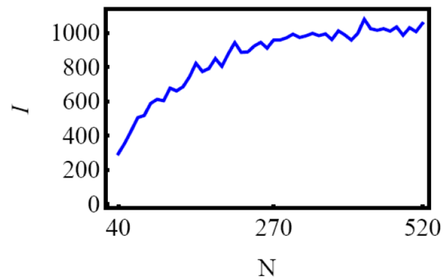

Fig. 3. The dependence of the peak intensity of radiation on the number of dipoles for the random distribution of initial phases in the interval [$-\pi$, π].

To clarify the cause of the discrepancies between the quantum Dicke model and the classical model, we simplify the model of Ref [51]. by using the mean field (MF) approximation. This allows us to reveal the mechanism of the SR burst.

3. Fluctuation theory of classical superradiance

The average value of the dipole $\langle c \rangle = N^{-1}\sum_{k=1}^{N} c_k$, in the right-hand side of Eq. (15), determines the MF affecting all dipoles. For the random distribution of initial phases of dipoles, one can expect that $\langle c \rangle = 0$. However, since we deal with a finite number of dipoles, *N*, the average dipole moment is never zero. Thus, dipoles are always in a nonzero MF. If we choose the regular uniform distribution of initial phases, then $\langle c \rangle = 0$ and an SR pulse does not arise. Therefore, a fluctuation of the initial phase distribution, which creates $\langle c \rangle \neq 0$, is necessary for SR.

Initially, while an SR pulse is not developed, the radiation intensity is proportional to the number of particles N, with $|\langle c \rangle| \sim 1/\sqrt{N}$, and the radiation time is the same as that of a single dipole. All the quantities can be considered constant at the timescale $\tau << \tau_0$ that we are interested in. During the SR pulse, the characteristic time of change of all quantities including the MF, $\langle c \rangle$, decreases by the factor of N. During this short interval of time, a number ΔN of dipoles participating in the SR burst loses their energy, and their contributions to the MF, $\langle c \rangle$, vanish. Nonetheless, we begin with considering a simplified MF model in which $\langle c \rangle$ does not change. Later, we check the obtained results by using a full description in which $\langle c \rangle$ evolves with time.

After $\langle c \rangle$ is replaced by a constant, $\langle c \rangle = E$, the emitters become independent, and the dynamics of each of them is described by the equation

$$\frac{dc}{d\tau} + i\theta\left(|c|^2 - 1\right)c = -E, \tag{15}$$

where E is a complex-valued constant corresponding to the value of the MF. This constant is equal to the average initial values of amplitudes $c_k(0)$: $E = N^{-1}\sum_{k=1}^{N} c_k(0)$ with $c_k(0) \sim e^{i\varphi_k}$. The constant E depends on the total number of emitters. In our computer simulations we use $N = 250$.

Since we are interested in the physics of the formation of an SR peak, we investigate the dynamics of the systems for the timescale $\tau < \tau_0$ during which the emitters are not yet in phase. For this timescale, $E(\tau) = N^{-1}\sum_{k=1}^{N} c_k(\tau) \approx 1/\sqrt{N} \approx E(0)$ and, therefore, the MF approximation in the form of Eq. (16) is applicable. As the comparison with numerical simulations shows, this model can be used for a description of the mechanism of the emitter phasing. Of course, the MF theory that we develop breaks down when emitters begin to phase and an SR burst arises.

The description of the system by the MF model is qualitatively correct. If the initial distribution of phases is uniform, the exact model [51] predicts $E = 0$ and no SR. In the MF model, if we set $E = 0$, we also obtain that there is no SR. On the other hand, for $E \neq 0$ even for the uniform distribution of initial phases, an SR pulse is observed (see Fig. 4).

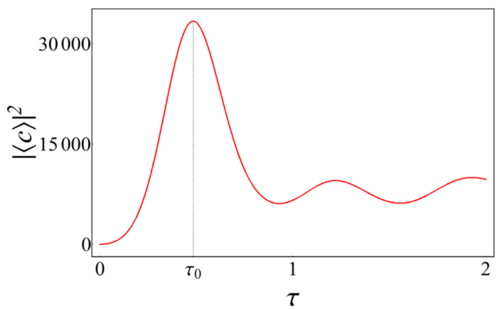

Fig. 4. The intensity of radiation of the system of non-interacting dipoles (the MF model) with $E = 0.1 \neq 0$ and the uniform distribution of initial phases. The delay time of the SR burst is $\tau_0 \approx 0.4$.

However, the intensity of SR radiation is not proportional to N^2. The reason for this is that only a fraction of all emitters participates in SR (see Fig. 5). In the MF model, for the regular uniform distribution of initial phases, there are neither fluctuations in the phase distribution nor interactions between emitters. The SR burst arises because there are emitters with characteristic initial phases that form an SR pulse. These characteristic phases are properties of a single independent emitter only. Therefore, to understand the origin of SR, we have to investigate how the dynamics of a single emitter depends on its initial phase.

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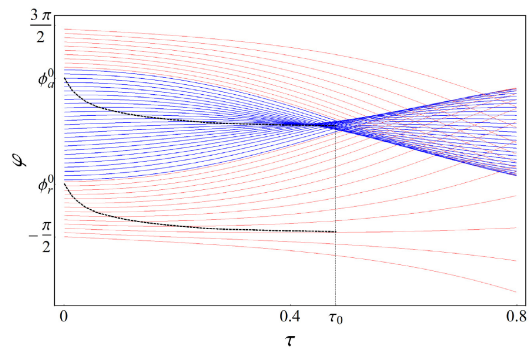

Fig. 5. The dynamics of phase oscillations of the envelopes for non-interacting dipoles (the MF model) with $E = 0.1$. Dotted lines show trajectories of ϕ_a and ϕ_r. The delay time of the SR burst is $\tau_0 \approx 0.4$. The trajectories of dipole phases participating in the fluctuation forming SA are shown by blue solid lines.

The dynamics of each dipole varies due to differences in dipole initial conditions. These differences and nonlinearity result not only in a simple phase shift but in a more complicated behavior. Indeed, Eq. (10) is the Duffing equation whose solution may be high frequency oscillations modulated by a low-frequency envelope [57]. In this case, Eq. (16) describing the envelope evolution has a periodic solution with a period determined by initial conditions. This is illustrated in Fig. 6 which shows that periods of oscillations of dipoles depend on initial phases.

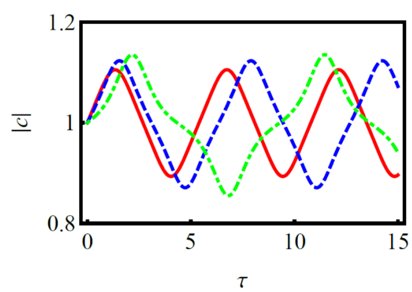

Fig. 6. Dependencies of periods of dipole oscillations on time. All dipoles have the same initial amplitude but different phases: Arg(c) = 0 (the red line), Arg(c) = π/10 (the blue line), and Arg(c) = π/5 (the green line). The dipoles are at the same MF, $E = 0.1$, $\tau = t / \tau_N$.

Here we note that, as one can see from Fig. 5, the characteristic time of the SR burst is $\tau_0 \approx 0.5$. At the same time, the characteristic period of slow oscillation envelopes is ≈ 5 (see Fig. 6). This means that during the SR burst, phases of emitters with different initial phases change weakly. Therefore, only a fraction of emitters, which phases get coordinated, participates in the SR burst. As we show below [see Eq. (18)], only emitters which phases belong to a characteristic interval $\Delta\phi$ participate in SR.

To investigate an effect of the distribution of initial dipole phases on SR, we represent amplitudes of envelopes and the field as $c_i(\tau) = |c_i(\tau)| \exp(i\varphi_i(\tau))$ and $E = E_0 \exp(i\psi)$, respectively. Now, Eq. (16) can be reduced to the equation for the phase dynamics:

$$\frac{d\varphi_i}{d\tau} + \theta\left(|c_i(\tau)|^2 - 1\right) = E_0 \sin(\varphi_i - \psi) / |c_i(\tau)|. \tag{16}$$

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At some values of $\varphi_i(0)$, the phase derivative in Eq. (17) turns to zero. Since all $|c_i(0)|^2 = 1$, to find these values of $\varphi_i(0)$ we have to solve the equation $E_0 \sin\phi_i = 0$ with $\phi_i = \varphi_i - \psi$. Near the solution of this equation, $\phi_i = 0$, at $\tau = 0$, the time derivatives of ϕ_i are negative for $\phi_i < 0$ and positive for $\phi_i > 0$. Thus, the trajectories $\phi_i(\tau)$ diverge from the point $\phi_i = 0$. Near another solution, $\phi_i = \pi + 2\pi n,\ n = 0, \pm 1, \ldots$, the situation is opposite so that the trajectories converge towards $\phi_i = \pi$. The dependencies of $\partial\phi_i / \partial\tau$ on the initial phase ϕ_0 and ϕ_i on time shown in Fig. 7 illustrate our arguments.

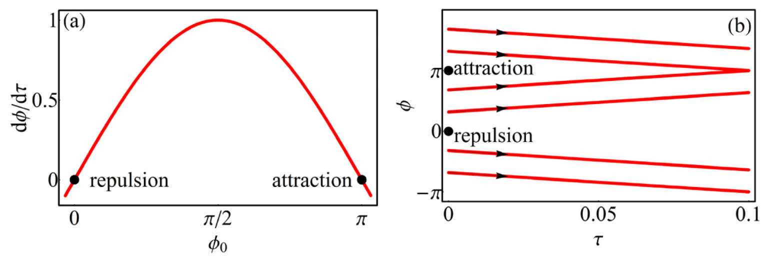

Fig. 7. (a) The dependence of the rate of change of ϕ_i on the initial phase ϕ_0 and (b) the phase dynamics for small times.

Thus, the initial distribution of dipole phases determines the phase ψ of the MF, which in turn, determines positions of attraction and repulsion points, $\varphi_a = \psi + \pi + 2\pi n,\quad n = 0, \pm 1, \ldots$ and $\varphi_r = \psi$, in the initial moment of time, $\tau = 0$ (see Fig. 7). These results allows us to evaluate the size of fluctuation forming an SR burst as $\Delta\phi_i \approx \phi_a - \phi_r = \pi$.

Now, let us model the exact system. The results of computer simulations for the time dependence of the phase shown in Fig. 8 demonstrate the presence of attraction and repulsion points corresponding to different values of the initial phase. The phases of envelopes of the dipole oscillations originating near the attraction point draw close forming a "time speckle" near $\phi_a \approx 0.4\pi$. At this moment, constructive interference in these envelopes occurs. Moreover, high-frequency oscillations interfere constructively as well because, as noted above, all fast oscillations are in phase.

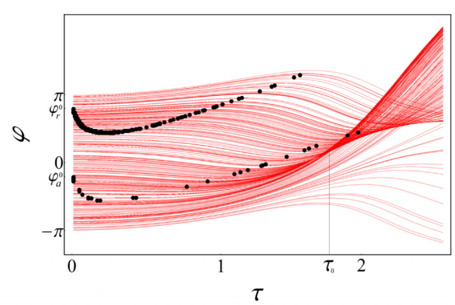

Fig. 8. The dynamics of phase oscillations of the envelopes for the exact model, $\varphi_r^0 \equiv \varphi_r(0) = \psi \approx 0.9\pi$, $\varphi_a^0 \equiv \varphi_a(0) = \psi - \pi \approx -0.1\pi$. Dotted line are trajectories of φ_a and φ_r which are terminated at the points $\varphi_a \approx 0.4\pi$ and $\varphi_r \approx 1.1\pi$ at the moment of time $\tau_0 \approx 1.9$ at which the SR burst occurs.

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The existence of condensation points in the initial moment $\tau = 0$ does not guarantee their presence for $\tau > 0$. However, computer simulation shows that these points do exist in later times. In Fig. 8, the dynamics of repulsion, φ_r, and attarction, φ_a, points is shown. As discussed above, in the initial moment of time, $\varphi_r^0 \equiv \varphi_r(0) = \psi$ and $\varphi_a^0 \equiv \varphi_a(0) = \psi - \pi$. Then, these points change their positions. For an SR pulse to arise, the existence of the repulsion and attraction points is essential while their positions is not important.

A comparison of Figs. 1 and 8 shows that the phase condensations and SR peaks occur at the same time, $\tau_0 \approx 1.9$. This leads to the conclusion that SR pulses arise due to the constructive interference in the envelopes $c_i(\tau)$ of dipoles which initial phases belong to the interval $\Delta\varphi = \Delta\phi$. The lifetime of the phase fluctuation is of the order of the period of envelope oscillations, τ_{env}, which is much greater than the main period of dipole oscillations. In Fig. 6, $\tau_{env} \sim 5$ while the period of dipole oscillations is $\tau_d \sim 1/(\omega\tau_N) \approx 10^{-3}$. During τ_{env}, emitters have enough time to superradiate and to lose their energy. We emphasize that not all dipoles take part in the SR pulse shown in Fig. 5. Obviously, the phenomenon is more pronounced if the number of dipoles with initial phases in the interval $\Delta\phi$ increases.

Let us estimate the size of the fluctuation forming an SR peak. During the time τ_0, when this peak is formed, the phases of a maximum number of dipoles must converge. Since $A_i \sim 1$, then from Eq. (17) one can obtain an estimate for an "optimal" size of the initial phase fluctuation

$$\Delta\phi \sim E_0\tau_0. \tag{17}$$

For a greater value of $\Delta\varphi$, phases do not have enough time to converge. For smaller $\Delta\varphi$, the number of dipoles taking part in the SR decreases. Fluctuations much smaller than the optimal one do not affect the system dynamics because the energy that they radiate is small and the time of radiation is much longer than the lifetime of the SR fluctuation. The numerical calculations show that the intensity of the radiation, $I = \left|\sum_i c_i\right|^2$, depends quadratically on the number of dipoles participating in the SR fluctuation (see Fig. 9).

Let us now estimate the delay time of the first SR peak. In both Dicke [8] and Vainshtein-Kleev [51] models, the dimensionless delay, τ_0, is proportional to $\log N$. In the MF model described here, the delay time weakly depends on the number of dipoles forming the SR fluctuation. As a result, the delay time does not depend on the number of dipoles in the system. However, one needs to take into account that in, our model, the MF depends on the number of dipoles in the system. The numerical experiment shows (Fig. 10) that the delay time depends on the MF as $\tau_0 \sim (E_0)^{-\alpha}$, $\alpha \approx 0.5$. Thus, the greater the MF is, the faster emitters are phased. Assuming that $E_0 \sim (N)^{-1/2}$ we obtain $\tau_0 \sim (N)^{-\alpha/2} \sim N^{0.25}$. This dependence is close to logarithmic.

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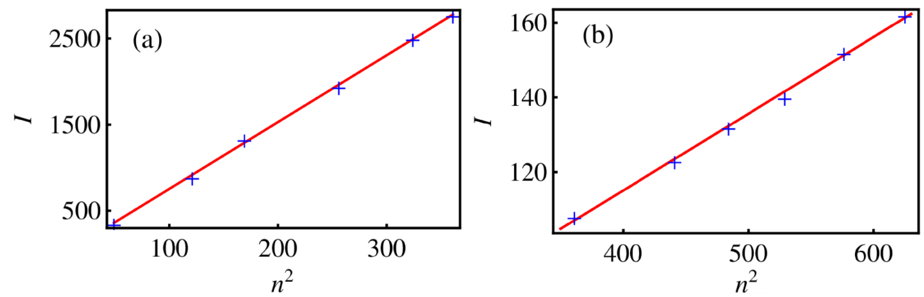

Fig. 9 . The dependence of the oscillation intensity on the squared number of dipoles forming the SR fluctuation in (*a*) the MF model and (*b*) the exact model.

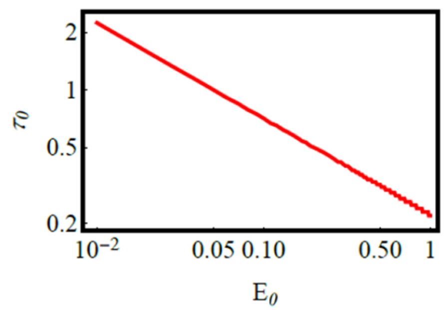

Fig. 10. The dependence of the delay time τ_0 on the MF E_0.

In summary, the MF model shows that there is an "optimal" number of dipoles participating in forming an SR peak. Initially, the phases of these dipoles are spread in the interval defined by Eq. (18). As a result, the time delay of the SR peak does not depend on the total number of dipoles. In the exact model, this dependence is weak (logarithmic) which may be related to the probability of forming the optimal fluctuation.

Thus, the MF model reflects the main feature the exact model. If the initial phase distribution is uniform, then the field of the radiative response is zero, and there is no SR. However, if there is a fluctuation in the initial phase distribution, then the field of the radiative response is nonzero leading to oscillating envelopes and their constructive inference forming an SR burst.

It is expected that there should be an optimal size $\Delta\varphi$ of the fluctuation that leads to the maximum intensity in the SR burst. The MF model predicts $\Delta\varphi = \pi$. To evaluate the real size of the fluctuation, we study the system with the uniform distribution of the initial phases which is disturbed by an artificially made fluctuation. We start with N_1 dipoles with the uniform phase distribution in the interval from 0 to 2π . To this set we add N_2 dipoles which phases are uniformly distributed in a smaller interval Δ. Note that this fluctuation forms the attraction point φ_a . Now, we check how the parameters Δ and N_2 affect the SR peak. The value of ψ should be inside the interval Δ.

We vary parameter Δ keeping the fluctuation size as $N_2 / N_1 = 0.1$. The results of numerical simulations are shown in Fig. 11 confirm that that there is an optimum width Δ at which the SR effect is the most pronounced.

From Fig. 6, one can expect that constructive interference of all emitters occurs after the time interval much greater than the oscillation period of $c(t)$. However, since only emitters that belong to the interval Δ, which is a small fraction of all emitters, interfere constructively, the delay time of an SR burst is much smaller than this period. Moreover, there exists an

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optimal fluctuation that gives an SR burst of the maximum intensity. As one can see from Fig. 11, an increase of Δ does not lead to an increase in the intensity.

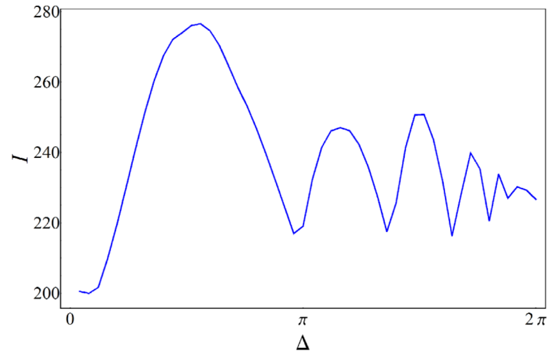

Fig. 11. The intensity of the first SR peak as a function of the width, Δ, of the fluctuation.

4. Conclusions

The suggested mechanism for the formation of an SR burst is based on the possibility of a fluctuation in the distribution of initial phases of dipoles. If phases of nonlinear dipoles are distributed uniformly in the interval $(-\pi,\pi)$, i.e. the number of dipoles $dN(\varphi)$ having the phase φ in the interval $d\varphi$ is $Nd\varphi/2\pi$, then SR does not occur. If there is a finite interval of phases (φ_1,φ_1) in which the number of dipoles $N_{\varphi_1\varphi_2}$ is greater than $N(\varphi_2-\varphi_1)/2\pi$, then over the time $t_0 \sim \ln N_{\varphi_1\varphi_2}$, an SR peak with the intensity $\sim N^2_{\varphi_1\varphi_2}$ will arise. In particular, this explains the SR peak in a system of dipoles with a random distribution of initial phases. Thus, the SR observed earlier in numerical experiments on the dynamics of a subwavelength array of classical nonlinear dipoles [51] is a purely classical phenomenon. It is not necessary to assume that emitters are identical. In our study, we assume that all dipoles have the same frequencies and coupling constants but we do not assume that they have the same phases. In contrast to quantum mechanical results, in classical physics, the evolution of each emitter, its phase and amplitude change, can be traced. Note that for linear oscillators the proposed mechanism would not work because the dipole phase depends linearly on an external force. If a phase fluctuation arises, its lifetime would be of the same order as the period of fast oscillations of an emitter. This cannot result in noticeable radiation of the field.

It is worth emphasizing that in Ref [53]. the validity of the MF approximation was questioned because the linear approximation cannot be applied to large amplitudes. The linear analysis only allows one to predict an exponential instability of the unperturbed solution. To elucidate the mechanism of phasing, nonlinear terms have to be taken into account. For this purpose, in our study, the MF approximation is applied without the linearization of the equations of oscillations.

Thus, SR arises as a result of the low-frequency modulation of oscillations of a nonlinear dipole acted upon by the field of its neighbors. The frequency of the modulation depends on both the initial phases of the oscillator and the near-field of radiation. The SR peak is the result of the constructive interference in slow envelopes of the dipole oscillations. SR arises when dipole phases coincide. The duration of the SR burst is determined by the frequency of the envelopes of the fast dipole oscillations.

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