Single-parameter scaling in one-dimensional Anderson localization: Exact analytical solution

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The variance of the Lyapunov exponent is calculated exactly in the one-dimensional Anderson model with random site energies distributed according to the Cauchy distribution. We derive an exact analytical criterion for the validity of single-parameter scaling in this model. According to this criterion, states with energies within the conduction band of the underlying nonrandom system satisfy single-parameter scaling when the disorder is small enough. At the same time, single-parameter scaling is not valid for states close to band boundaries and those outside of the original spectrum, even in the case of small disorder. The results obtained are applied to the Kronig-Penney model with the potential in the form of periodically positioned δ functions with random strengths. We show that an increase in disorder can restore single-parameter scaling behavior for states within the band gaps.

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I. INTRODUCTION

The hypothesis of single-parameter scaling (SPS) in the context of transport properties of disordered conductors was introduced in Ref. 1. It was suggested that scaling properties of the conductance g are determined by a single parameter, the conductance itself, through a scaling equation

$$\frac{d(\ln g)}{d(\ln L)} = \beta(g), \tag{1}$$

where L is the size of a sample. The nature of the scaling conductance g was debated for some time until it was understood[2] that scaling in the theory of localization must be interpreted in terms of the entire distribution function of conductivity rather than in terms of its momentums. SPS in this case means that the distribution function of g is fully determined by a single parameter, which obeys the scaling equation (1). In Ref. 2, which was concerned with scaling properties of one-dimensional disordered conductors, Anderson *et al.* proposed the parameter

$$\tilde{\gamma}(L) = \frac{1}{2L}\ln\left(1+\frac{1}{g}\right) \tag{2}$$

as a scaling parameter suitable to describe the fluctuations of the conductivity. In the limit $L\to\infty$ the introduced parameter takes a nonrandom value γ, which is the inverse localization length l_{loc} or the Lyapunov exponent (LE) characterizing the spatial distribution of electron's wave functions.[3,4] It was suggested in Ref. 2 that the introduced parameter has a normal distribution and does not exhibit anomalously large fluctuations. Calculations carried out in Ref. 2 showed that variance of the LE, σ^2, scales according to the law of large numbers, $\sigma^2 \sim 1/L$, and is related in a universal way to LE (Ref. 5):

$$\sigma^2 = \gamma/L. \tag{3}$$

This relation is the essence of SPS in the case of strong localization, as it presumes that two parameters of the normal distribution of the LE are reduced to one scaling parameter γ. According to the scaling theory (see Ref. 1 and references therein) almost all states in one-dimensional systems are localized, and these systems are, therefore, always in the regime of strong localization in the asymptotic limit $L \gg l_{loc}$.

Equation (3) was first derived in Ref. 2 within the approximation known as the random phase hypothesis, which assumes that there exists a microscopic length scale over which phases of complex transmission and reflection coefficients become completely randomized. Under similar assumptions, Eq. (3) was rederived later by several authors for a number of different models.[6–11]

Landauer's representation of the conductance in terms of transmission coefficients for different scattering channels[12,13] reduces the study of the conductance in quasi-one-dimensional wires to the analysis of scattering or transfer matrices. Within the transfer-matrix approach[14] the problem is further reduced to the study of statistical properties of the products of random matrices (see Ref. 15 and references therein). In this context, the self-averaging of LE and its normal distribution in the asymptotic limit $L\to\infty$ are rigorously established mathematical facts.[3,16,17] SPS expressed by Eq. (3) was also established in the limit of strong localization for a quasi-one-dimensional geometry in Refs. 18 and 19 with the use of the Dorokhov-Mello-Pereyra-Kumar equation.[20]

The SPS hypothesis has also been verified in the regime of weak localization, which exists in the conducting phase of three-dimensional conductors and in the limit of a large number of scattering channels in the quasi-one-dimensional geometry. In contrast with the case of strong localization, the distribution of the conductance (rather than the logarithm of the conductance) has Gaussian form with two independent parameters. The variance of the conductance, however, was found to be a universal number,[21,22] leaving one again with a

single scaling parameter: the average conductance. The universal conductance fluctuations, which were first discovered in three-dimensional conductors within a diagrammatic approach,[21,22] were later reinterpreted from the point of view of the random matrix theory in a quasi-one-dimensional geometry (see for review Ref. 15).

The scaling properties of conductivity have also been studied numerically by a number of authors. The general concepts of the scaling theory expressed by Eq. (1) were verified by means of Green's function and transfer-matrix approaches generalized for two- and three-dimensional systems in Ref. 23. The log-normal distribution of the conductance and SPS, Eq. (3), has been confirmed for the Anderson model (AM) in numerical simulations in Ref. 19. In the one-dimensional situation, the existence of SPS has been also obtained in simulations of AM with correlated disorder[24] and scalar wave propagation in superlattices with different models of randomness.[25,26]

The zero energy state in one-dimensional models with off-diagonal disorder (random hopping models) represents a special case. These models demonstrate a delocalization transition in the vicinity of zero energy[27] contrary to the conclusion of the scaling theory that such a transition is absent in one-dimensional systems. The SPS relation (3) between the standard deviation of the LE and its mean value is also violated in this case.[28] Unusual properties of this model are due to a so-called chiral symmetry, which is characteristic of the state with $E=0$ in this model. In further discussion we will ignore this special case and refer to regular situations, which include models with diagonal disorder and random hopping models away from the critical $E=0$ point.

Simultaneously with numerous confirmations of the existence of SPS, the limits of its validity have been the subject of intensive discussions (see, for example, Ref. 29 and references therein). As we mentioned above, the original condition for SPS, postulated in Ref. 2, invokes the hypothesis of the phase randomization. This hypothesis implies that the phases of complex transmission and reflection coefficients become completely randomized at the distances much smaller than the localization length. Phase randomization was numerically studied for the AM in Ref. 30, where it was shown that for small disorder the phases indeed become uniformly distributed at a scale much shorter than the localization length. This does not happen, however, for states in the center of the original conduction band. Numerical calculation of the AM in Ref. 30 and analytical consideration of a model with periodically positioned scatterers in Ref. 31 obtained a nonuniform phase distribution for such states. It was shown in Ref. 31 that for the states at the center of the band there exists a new length, a phase relaxation length l_φ. As soon as the length of the sample exceeds l_φ, the phase distribution approaches a stationary but nonuniform form. Under certain conditions, the relaxation length rather than the mean free path was found to determine the localization length. At the same time, neither analytical nor numerical studies of the states at the center of the conduction band found violations of SPS. These results cast doubt upon the relevance of phase randomization for SPS. In this paper we show that the condition for SPS is not phase randomization.

When local disorder is strong, the phase distribution was found to never become uniform,[30] and the probability distribution of LE, in this case, is controlled by two independent parameters.[29,30] It was indicated,[30] however, that even in the case of an extremely nonuniform phase distribution, the deviations from SPS are rather limited.

The hypothesis of phase randomization lies at the foundation of all existing theoretical approaches to statistical properties of conductance, including those based upon random matrix theory.[15] An additional requirement crucial for Eq. (3) can be called ''local weakness of disorder.'' In calculations based upon the random matrix theory,[17,20] this requirement is set as a limit when the cross section of each individual scatterer tends to zero, while the density of the scatterers tends to infinity keeping the localization length constant. It is commonly believed that in the regime of strong localization ($l_{loc} \ll L$), SPS holds provided that the local disorder is weak, so that the localization length exceeds all microscopic length scales of the model. Increase of the disorder leads to reduction of the localization length, and eventually violates SPS.

Results which apparently contradict this well established understanding of the crossover between SPS and statistics with two independent parameters were recently reported in Ref. 32. The system considered in Ref. 32 belongs to the class of Kronig-Penney-like models (KPM's), which have been intensively studied (see, for example, Refs. 4, 33 and 34 and references therein). The original spectrum of KPM's contains multiple bands separated by band gaps. Disorder not only localizes states within the original pass bands but also creates tails of localized states in former band gaps.[4] According to Ref. 32 the spectrum of the system is divided into two groups of states with different scaling behavior: SPS holds for states from the conduction bands of the initial spectrum and is violated for states from initial band gaps. Moreover, this violation of SPS for band gap states occurs even for weak disorder and turns out to be much more dramatic than the phase randomization approach would predict.[30]

The occurrence of states outside of the initial conduction bands is known to be a model independent phenomenon. It seems plausible, therefore, that the coexistence of SPS and non-SPS states found in Ref. 32 is not a particular feature of KPM's but rather a general property of quantum disordered systems.

The main objective of the present paper is to reexamine the problem of scaling properties of conductance in one-dimensional systems and to derive SPS, Eq. (3), without the assumption of phase randomization. This calculation allows us to formulate a ''correct'' criterion for SPS and to understand the nature of its violation reported in Ref. 32. The main results of this paper were outlined in Ref. 35.

The paper is organized as follows: In Sec. II we formulate the model within which we calculate the variance of LE. The details of the calculations are presented in Sec. III. The new criterion of SPS is derived and analyzed in Sec. IV. In that section we also complement our analytical calculations with numerical simulations of a more generic model. Comparison with the latter helps us to distinguish between universal features of our results and those specific to the selected model.

The transition between SPS and non-SPS states is discussed in Sec. V. We conclude in Sec. VI.

II. DESCRIPTION OF THE MODEL

Let us consider a one-dimensional tight-binding model with diagonal disorder, which is described by the following equations of motion:

$$\psi_{n+1}+\psi_{n-1}-U_n\psi_n=0, \tag{4}$$

where ψ_n represents the wave function of the system at the nth site. In Eq. (4) the hopping integral is chosen to be equal to 1, so it sets the energy scale in the system. The concrete meaning of U_n depends upon the interpretation of the model (4). There are two apparently different models that can be described by Eq. (4). In the first, this equation represent a classical AM, with U_n defined as

$$U_n=-E+\epsilon_n, \tag{5}$$

where E is the energy of a particle and ϵ_n is the random site energy. Second, it can be shown (see, for example, Ref. 4) that the Schrödinger equation for KPM's with a random potential formed by periodically positioned δ functions with random strengths, V_n, also reduces to the form (4) with ψ_n being the values of the eigenfunctions at the sites occupied by the δ potentials. In this case, U_n is defined as follows[4]:

$$U_n=2\cos(ka)+\frac{V_n}{k}\sin(ka), \tag{6}$$

where $k=\sqrt{E}$ is the energy variable and a is the period of the structure. To be able to obtain an exact analytical solution, we assume that parameters ϵ_n or V_n are distributed with the Cauchy probability density (the Lloyd model[36])

$$P_C(x)=\frac{1}{\pi}\frac{\Gamma}{\Gamma^2+(x-x_0)^2}, \tag{7}$$

where $x_0=0$ or V_0 for the AM or KPM's, respectively. Parameters x_0 and Γ represent the mean value of the random variable x and the width of the distribution, respectively. Although Γ characterizes the strength of disorder in the system, it cannot be interpreted as a second moment of the distribution (7), because the latter does not exist. The AM with the probability distribution (7) is one of the first models where the LE was evaluated exactly.[4,37] The probability distribution of parameters U_n, which enter equations of motion (4), has the same form as Eq. (7) with the following parameters:

$$\langle U_n\rangle=U_0=\begin{cases}2\cos(ka)+\dfrac{V_0}{k}\sin(ka), & \text{KPM},\\ E, & \text{AM},\end{cases} \tag{8}$$

$$\Gamma_U=\begin{cases}\dfrac{\Gamma}{k}|\sin(ka)|, & \text{KPM},\\ \Gamma. & \text{AM}.\end{cases} \tag{9}$$

In the absence of disorder, the energy spectrum of the model is determined by the condition: $|U_0|<2$. In the AM this leads to a single conduction band $-2\leqslant E\leqslant 2$. In KPM's there exist multiple bands separated by band gaps. Allowed values of the energy variable belong to intervals

$$k_n^b a<ka<\pi n, \quad n=1,2,3\ldots, \tag{10}$$

where k_n^b obeys the equation

$$\tan\left(\frac{k_n^b a}{2}\right)=\frac{V_0}{2k_n^b}, \quad n \text{ odd}, \tag{11}$$

$$\tan\left(\frac{k_n^b a}{2}\right)=-\frac{2k_n^b}{V_0}, \quad n \text{ even}. \tag{12}$$

The higher-energy boundaries of each band correspond to so-called resonances[4] because disorder does not affect transport at these particular energies. This fact can easily be seen from Eq. (9), where Γ_U for KPM's becomes zero for all $ka=\pi n$. The presence of these resonances is a specific property of the model under consideration caused by the strict periodicity in the positions of site potentials. Similar resonances are also present in other models such as the dimer model[38] or models of random superlattices.[25,32] The resonances dissappear once one destroys the exact periodicity in the positions of δ functions or allows for random variations in the width of superlattice's layers.

The main objects of our study are the finite-size LE $\tilde{\gamma}(L)$ and its variance σ^2. Here $\tilde{\gamma}(L)$ can be defined for the model under consideration as[4]

$$\tilde{\gamma}(L)=\frac{1}{L}\ln r_N, \tag{13}$$

where $N=L/a$ is the total number of sites in the system and r_N is the envelope of the wave function:

$$r_N=(\psi_N^2+\psi_{N-1}^2)^{1/2}. \tag{14}$$

As we discussed in the Introduction, $\tilde{\gamma}(L)$ takes a nonrandom value, γ, in the limit $L\to\infty$. This limiting value can also be considered as an average of $\tilde{\gamma}(L)$ over different realizations of the system.[3,4] For large but finite L, $\tilde{\gamma}(L)$ exhibits finite-size fluctuations whose distribution function asymptotically approaches the Gaussian form with the variance σ^2 decreasing as $1/L$.[4,16,17]

The average LE γ in the considered model was first calculated in Ref. 37. It turns out that the method developed in that paper (see also Ref. 4) can be as well used for exact calculation of σ^2. The method is based upon the representation of LE in terms of the phase variable z_n, defined as $z_n=\psi_n/\psi_{n-1}$, which obeys the following equation of motion:

$$z_n + z_{n-1}^{-1} = U_n . \tag{15}$$

The finite-size LE can be expressed in terms of z_n:

$$\tilde{\gamma}(L) = \frac{1}{L}\sum_{n=1}^{N} \ln|z_n| + \frac{1}{2L}\ln\left(r_0^2 \frac{1+z_{N+1}^2}{z_0^2}\right). \tag{16}$$

If z_n is a stationary random function of n—that is, a distribution of z_n is independent of n—the first term in Eq. (16) is of the order of unity while the second term is of the order of $(1/L)$ and disappears in the limit $L\to\infty$. The expression for the LE, therefore, takes the following form:

$$\gamma = \lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{N} \ln|z_n| = \langle \ln|z_n|\rangle, \tag{17}$$

where the average on the right-hand side is taken over the stationary distribution of z.

The asymptotic expression for variance of the LE can be obtained from Eq. (16),

$$\sigma^2 = \frac{1}{L^2}\sum_{m,n=1}^{N} [\langle \ln z_m \ln z_n\rangle - \langle \ln z_m\rangle\langle \ln z_n\rangle], \tag{18}$$

and is valid as long as the system's size L is much greater than the correlation radius of z_n, which we assume to be finite.

III. VARIANCE OF THE LYAPUNOV EXPONENT IN THE LLOYD MODEL

A. Two-point distribution of the phases z_n

Calculation of the variance from Eq. (18) requires knowledge of the two-point distribution function $P_2(z_n, z_m)$ of the phases z. Our calculations of this function are based upon representation of a joint distribution of multiple random variables as the product of marginal and conditional distributions:

$$P_2(z_n, z_{n+k}) = P_1(z_n) P(z_n|z_{n+k}), \tag{19}$$

where $P_1(z_n)$ is a stationary probability distribution of z_n and $P(z_n|z_{n+k})$ denotes a conditional probability distribution of z_{n+k} provided that z_n is fixed. With the help of Eq. (15) the latter probability can be written as

$$P(z_n|z_{n+k}) = \int \delta(z_{n+k} + z_{n+k-1}^{-1} - U_{n+k-1}) \times P(z_n|z_{n+k-1}, U_{n+k-1}) dU_{n+k-1} dz_{n+k-1},$$

where $P(z_n|z_{n+k-1}, U_{n+k-1})$ is a joint probability of z_{n+k-1} and U_{n+k-1}. It follows from the structure of Eq. (15) that z_n depends only upon values of the random parameter U_m at preceding sites $m<n$ and thus is independent of U_n. The joint probability $P(z_n|z_{n+k-1}, U_{n+k-1})$, therefore, can be factorized and integration over U_{n+k-1} can be carried out. The result is the following recurrent relation between $P(z_n|z_{n+k})$ and $P(z_n|z_{n+k-1})$:

$$P(z_n|z_{n+k}) = \int P(z_n|z_{n+k-1}) P_C(z_{n+k} + z_{n+k-1}^{-1}) dz_{n+k-1}, \tag{20}$$

where P_C is the Cauchy distribution introduced in Eq. (7). The advantage of the Cauchy distribution is that recurrence (20) can be solved exactly. The conditional probability obtained has again the form of the Cauchy distribution, which can be conveniently presented in the form

$$P(z_n|z_{n+k}) = \frac{\mathrm{Im}\,\xi_k}{\pi}\frac{1}{(z_{n+k}-\xi_k)(z_{n+k}-\xi_k^*)}, \tag{21}$$

where the asterisk denotes complex conjugation and parameters ξ_k obey the following equation:

$$\xi_k + \xi_{k-1}^{-1} = U_0 + i\Gamma. \tag{22}$$

Equation (21) for $P(z_n|z_{n+k})$ and Eq. (22) for ξ_k have exactly the same form as those obtained in Ref. 37 for the one-point distribution $P_1(z_n)$. However, in the case of the one-point distribution one looks for a stationary solution of Eq. (22), while the conditional distribution $P(z_n|z_{n+k})$ requires that Eq. (22) be solved with the initial condition

$$\xi_0 = z_n . \tag{23}$$

This solution can be presented as

$$\xi_k = \frac{\delta^k - \delta^{-k} - z_n(\delta^{k+1} - \delta^{-k-1})}{\delta^{k-1} - \delta^{-k+1} - z_n(\delta^k - \delta^{-k})}, \tag{24}$$

where δ is the k-independent solution of Eq. (22), which obeys the stationary version of Eq. (22):

$$\delta + \delta^{-1} = U_0 + i\Gamma. \tag{25}$$

Real and imaginary parts of δ determine the center and the width of the one-point distribution $P_1(z_n)$:

$$P_1(z_n) = \frac{\mathrm{Im}\,\delta}{\pi}\frac{1}{(z_n-\delta)(z_n-\delta^*)}. \tag{26}$$

Averaging Eq. (17) with the probability distribution (26), one obtains the average LE (Refs. 4 and 37) γ,

$$\gamma = \ln|\delta|/a. \tag{27}$$

Equations (21), (24), and (26) determine the two-point probability distribution $P_2(z_n, z_{n+k})$ defined by Eq. (19).

B. Variance of the Lyapunov exponent: General expression

Equation (18) for the variance of the LE, σ^2, can be presented in the following form:

$$\sigma^2 = \frac{2}{L^2}\sum_{n=0}^{N-1}\sum_{k=1}^{N-n} \langle \ln|z_n| \ln|z_{n+k}|\rangle + \frac{1}{L}\langle \ln^2|z_n|\rangle - \gamma^2. \tag{28}$$

The correlation function $D(k)=\langle \ln|z_n|\ln|z_{n+k}|\rangle$ is independent of the initial site n. With the use of the probability distribution $P_2(z_n,z_{n+k})$ found in the previous subsection, it can be presented as

$$D(k)=\frac{\mathrm{Im}\,\delta}{\pi}\int_{-\infty}^{\infty}\frac{\ln|z|\ln|\xi_k(z)|}{(z-\delta)(z-\delta^*)}dz, \tag{29}$$

where $\xi_k(z)$ and δ are defined by Eqs. (24) and (25), respectively. Interchanging the order of integration and summation in Eq. (28) one can find for the variance

$$\sigma^2=\frac{2\,\mathrm{Im}\,\delta}{\pi aL}\int_{-\infty}^{\infty}\frac{\ln|z|\ln|z\delta-1|}{(z-\delta)(z-\delta^*)}dz - \frac{1}{aL}\varphi(\pi-\varphi)-\frac{2}{L}\gamma\,\ln(\delta^2-1)+O(1/L^2), \tag{30}$$

where φ is the phase of δ,

$$\delta=p\,\exp(i\varphi),$$

reduced to the interval $[0,\pi]$, and p denotes the absolute value of δ: $\delta=|p|$. The remaining integral in Eq. (30) can be further simplified with the use of an appropriate contour in the complex z plane. One, finally, arrives at the following expression for the variance in the case $\varphi<\pi/2$, which corresponds to $U_0>0$:

$$\sigma^2=\frac{1}{L}\left\{-\gamma\,\ln\left[2\frac{\cosh(2\gamma a)-\cos(2\varphi)}{\sinh^2(\gamma a)}\right] + \frac{1}{a}\int_{\varphi}^{\pi}dx\,\tan^{-1}\left[\frac{\sinh(2\gamma a)\sin\varphi}{\cosh(2\gamma a)\cos\varphi-\cos x}\right]\right\}+O(1/L^2). \tag{31}$$

Since our model is symmetric with respect to the transformation $\varphi\to\pi-\varphi$, the variance for $\varphi>\pi/2$ can be easily evaluated.

IV. NEW CRITERION FOR SINGLE-PARAMETER SCALING

The necessary (but not sufficient) condition for SPS to hold is that the localization length be greater than all microscopic scales in the system. Therefore, we should consider the general result, Eq. (31), in the limit of large localization length, $\gamma a\ll 1$. Our first goal is to develop an asymptotic form of the integral in Eq. (31) in this limit. This is not a trivial task since the integral has a singularity at $\gamma=0$. The first term in Eq. (31) is also singular at this point, and one would anticipate two singularities to cancel out. The latter singularity has a logarithmic nature, $\gamma\ln\gamma$, and we need, therefore, to extract the similar logarithmic singularity from the integral in Eq. (31). To this end, we first evaluate the integral in Eq. (31) by parts and present it in the following form:

$$\Phi(\gamma,\varphi)=\pi\tan^{-1}\left(\frac{\beta}{1+\zeta}\right)-\varphi\tan^{-1}\left(\frac{\beta\cos\varphi}{\zeta-\cos\varphi}\right) + \beta\int_{-1-\zeta}^{-(\zeta-\cos\varphi)}dx\,\frac{\arccos(x+\zeta)}{x^2+\beta^2}, \tag{32}$$

where parameters β and ζ are defined as

$$\beta=\sinh(2\gamma a)\sin\varphi, \tag{33}$$

$$\zeta=\cosh(2\gamma a)\cos\varphi. \tag{34}$$

Since we are interested in small values of γa, we can expand $\cos^{-1}(x+\zeta)$ from Eq. (32) in the Taylor's series in x,

$$\cos^{-1}(x+\zeta)=\cos^{-1}(\zeta)+\sum_{n=1}^{\infty}a_n x^n, \tag{35}$$

and carry out the termwise integration. As the result we obtain

$$\Phi(\gamma,\varphi)=\cos^{-1}(\zeta)\left\{\tan^{-1}\left(\frac{1+\zeta}{\beta}\right)-\tan^{-1}\left(\frac{\zeta-\cos\varphi)}{\beta}\right)\right\} - \frac{1}{2}\frac{\beta}{\sqrt{1-\zeta^2}}\ln\left[\frac{(\zeta-\cos\varphi)^2+\beta^2}{(1+\zeta)^2+\beta^2}\right] + \beta\{F[-(\zeta-\cos\varphi)]-F(-1-\zeta)\}, \tag{36}$$

where $F(x)$ is defined by the following power series:

$$F(x)=\sum_{n=2}^{\infty}a_n\frac{x^{n-1}}{n-1}.$$

Comparing this expression with the original series (35) one can obtain for $F(x)$ the following integral representation:

$$F(x)=\frac{1}{x}\cos^{-1}(\zeta)+\frac{\ln|x|}{\sqrt{1-\zeta^2}}+\int\frac{dx}{x^2}\cos^{-1}(\zeta+x). \tag{37}$$

The remaining integral in Eq. (37) can be calculated exactly. As a result we have

$$F(x)=\frac{1}{x}[\cos^{-1}(\zeta)-\cos^{-1}(\zeta+x)] + \frac{1}{\sqrt{1-\zeta^2}}\ln|1-\zeta^2(\zeta+x) + \sqrt{(1-\zeta^2)[1-(\zeta+x)^2]}|. \tag{38}$$

When one combines Eq. (31) with Eqs. (36) and (38), the logarithmic singularity in $\Phi(\gamma,\varphi)$ nicely cancels out the singularity in the first term of Eq. (31). The expression for the variance σ^2 emerging in the leading in γ order takes the SPS form

$$\sigma^2\simeq 2\gamma/L. \tag{39}$$

This is, to the best of our knowledge, the first truly microscopic derivation of SPS with no *ad hoc* hypotheses. The main reward for this is the exact criterion for SPS, which follows from the conditions under which we have arrived at Eq. (39). First of all we assumed that

$$\frac{\beta}{\sqrt{1-\zeta^2\cos^2\varphi}} \ll 1. \tag{40}$$

Since

$$\beta \simeq 2\gamma a \sin\varphi, \quad \zeta \simeq 1+2(\gamma a)^2, \tag{41}$$

the inequality (40) can be recast in the form

$$\gamma a \ll \sin\varphi. \tag{42}$$

Another condition, which we have to impose in order to obtain Eq. (39), is

$$(\zeta-1)\cos\varphi/\beta \simeq a\gamma/\tan\varphi \ll 1. \tag{43}$$

Since $\sin\varphi \leqslant \tan\varphi$, the first of the two inequalities is more restrictive, and the final condition for SPS takes the form

$$\kappa = l_{loc}/l_s \gg 1, \tag{44}$$

where $l_{loc}=\gamma^{-1}$ is the localization length and a new length l_s is defined as

$$l_s = a/\sin\varphi. \tag{45}$$

Evaluation of the integral in Eq. (31) in the limit $\kappa \ll 1$ can be performed by means of a simple expansion of the integrand in power series in β and retaining only the linear in β term. The resulting expression for σ can be presented as

$$\sigma^2 = \frac{1}{l_s L}\left(\pi - \frac{2l_{loc}}{l_s}\right). \tag{46}$$

Equation (46) shows that in the regime considered here, σ^2 is determined by the new length l_s rather than by l_{loc}. It is important to emphasize that in this limit both lengths l_s and l_{loc} can far exceed a, and Eq. (46), therefore, describes the violation of SPS while the system remains within a meaningful scaling regime.

It should be noted, however, that Eq. (39) differs from Eq. (3) by the factor of 2. This discrepancy is due to the peculiar nature of the Cauchy distribution, whose moments, starting from the second one, diverge. Because of this, none of the standard approaches, used to derive Eq. (3) within the random phase hypothesis, can be applied to the Lloyd model. In order to illustrate this point, let us consider, for example, an expression for σ^2 obtained in Ref. 30 for the AM:

$$\sigma^2 = \frac{1}{N}\left\{ \mathrm{Var}\left[\ln\left(1+\frac{\epsilon^2}{4-E^2}\right)\right] + \left\langle \left(\ln\left[1+2\frac{\epsilon}{\sqrt{4-E^2}}\cos\nu + \frac{\epsilon^2}{4-E^2}\right]\right)^2 \right\rangle \right\}. \tag{47}$$

Brackets $\langle\ \rangle$ designate here averaging over the random site energy ϵ and the phase ν, which is assumed to be statistically independent of ϵ and distributed uniformly. ''Var'' denotes the variance of the respective quantity. The standard weak disorder expansion used in Ref. 30 implies the expansion of this expression in powers of the random variable ϵ with consecutive averaging. The first term in Eq. (47) then becomes of the order of $(\langle\epsilon^2\rangle/[4-E^2])^2$ and is neglected, while the second term after averaging over the phase yields Eq. (3). In the case of the Cauchy distribution for ϵ, this approach cannot be applied because $\langle\epsilon^2\rangle$ does not exit. In order to pass to the weak-scattering limit one has to average over ϵ first, and only after that carry out expansion over the parameter Γ of the Cauchy distribution. Both terms in Eq. (47) then become of the same order of magnitude $\Gamma/\sqrt{4-E^2}$, and though the general proportionality between σ^2 and the LE is preserved, the numerical factor not equal to that of Eq. (3) or of Eq. (39). This result implies that the phase randomization hypothesis is not valid at all for the Lloyd model. What is more important, however, is the fact that although the phase randomization hypothesis fails, SPS still survives.

For $\kappa \ll 1$, one can provide a clear physical interpretation for the length l_s. According to Thouless,[39] the phase $\varphi(E)$ is proportional to the integrated density of states $G(E)=\varphi(E)/\pi$, and $\kappa \ll 1$ corresponds to either $G(E) \ll 1$ or $1-G(E) \ll 1$. The length l_s then can be expressed in terms of the number of states in the energy intervals between E and the closest boundary of the spectrum, $l_s = 1/[\pi G(E)]$ [or $l_s = a/(\pi - a\pi G)$]. For the AM these boundaries lie at $\pm\infty$, and for KPM's they are the resonance boundaries of the bands, where $\varphi(E)=\pi n$ with n an integer. The states in these regions arise due to rare realizations of the disorder, and can be associated with spatially localized and well-separated structural defects. The length l_s then can be interpreted as an average distance between such defects. In view of this interpretation of l_s, the physical meaning of the transition between two types of scaling regimes also becomes clear. Condition (44) means that the localization length at the energy E exceeds the spatial separation between neighboring localization centers from the relevant part of the tail (between E and the nearest boundary of the spectrum). Under this condition the localized states overlap and SPS is valid.

To complete our discussion of the new scaling parameter l_s, let us compare it with the phase randomization length, l_{ph} numerically studied by Stone *et al.*[30] Assuming that E is inside the conduction band and far from the band boundaries, we can approximate l_s as

$$l_s^{-1} \simeq \frac{1}{2a}\sqrt{4-U_0^2} + O(\Gamma_U^2). \tag{48}$$

According to Eq. (48), l_s decreases toward the center of the band $U_0=0$, where it reaches its minimum value equal to a. At the same time, the phase randomization length was found in Ref. 30 to increase toward the center of the band $E=0$, where it seemed to diverge. For $E=0$, the phase distribution was found to be nonuniform even for very long chains. The absence of the phase randomization in the center of the band was also found analytically in Ref. 31. This comparison

proves that l_s is an independent new parameter responsible for the statistics of the LE. Both numerical results of Ref. 30 and analytical calculations of Ref. 31 show that a nonuniform distribution of phases can be consistent with SPS, providing an additional argument against the condition for SPS based upon phase randomization. At the same time, our criterion, Eq. (44), correctly predicts validity of SPS in the band center as long as the localization length remains macroscopic.

l_s and the SPS criterion in generic models

The peculiarities of the Lloyd model may cast doubts on the robustness of the new scale l_s and the criterion, Eq. (44). In order to show that this criterion is applicable beyond the Lloyd model, we carry out additional numerical simulations of the model studied previously in Ref. 32. That model is of the Kronig-Penney kind, but unlike the model considered in the present paper, its potential is formed by rectangular barriers. The width of the barriers is assumed to be random with a uniform distribution over a given interval. Both the potential and the statistics of the model used in numerical simulations are considerably different from the Lloyd model; e.g., all moments of the distribution function exist.

It is instructive to rewrite expressions for σ^2 in terms of a new dimensionless variable $\tau=\sigma^2 L l_{loc}/2$ as a function of κ. In terms of these variables both asymptotics of the variance σ^2 given by Eqs. (39) and (46) can be presented in a form which contains no free parameters:

$$\tau=\begin{cases} \kappa\left(\dfrac{\pi}{2}-\kappa\right), & \kappa\ll 1, \\ 1, & \kappa\gg 1. \end{cases} \tag{49}$$

Although we do not expect the concrete form of the function $\tau(\kappa)$ to be universal, we do believe that the new crossover length l_s retains its physical meaning in the general case, and that the crossover point is also universally determined by $\kappa \sim 1$.

In order to generalize the crossover length l_s for other models, we use the interpretation of the phase φ in terms of the integral density of states normalized in such a way that the phase would vary between 0 and π for any given band. The generalization is quite straightforward for models with a single-band spectrum if in the absence of disorder the band has a finite width. The total number of states in such models is finite, and it can be used to normalize the phase. If the initial band of the system is infinitely broad, e.g., for the Schrödinger equation with a random white-noise potential, one has to introduce a cutoff frequency for the spectrum in order to normalize the phase. This cutoff introduces a microscopic length similar to the intersite distance a, which is used to define the localization length and the density of states. The crossover parameter κ then can be obtained from an expression relating the LE and the integral density of states found in Ref. 4:

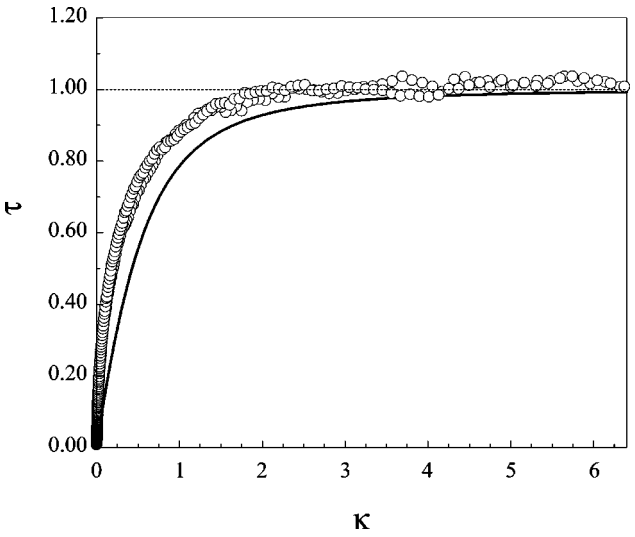

FIG. 1. The function $\tau(\kappa)$ obtained from the analytical solution of the Lloyd model, Eq. (31) (solid line), and the numerical simulations of KPM with rectangular barriers of random widths (open circles). Note that the SPS equations for these models differ by a factor of 2 [Eqs. (3) and (39)]; therefore we rescaled numerical data by this factor.

$$\kappa^{-1}(E)=\frac{\sqrt{\pi}}{2}\int_0^\infty \sqrt{x}\exp\left(-\frac{x^3}{12}-\frac{Ex}{D^{2/3}}\right)dx, \tag{50}$$

where D determines the strength of the δ-correlated potential. The asymptote of κ for large negative E is

$$\kappa\approx\exp\left(-\frac{4|E|^{3/2}}{3D}\right)\ll 1,$$

and we conclude that these states do not obey SPS. The transition to SPS behavior again occurs at the initial band boundary $E=0$, where $\kappa\approx 1.1$ and does not depend upon parameters of the model.

For systems with multiple (in the absence of disorder) bands separated by band gaps, one has to consider separately two different situations. If disorder-modified bands still do not overlap and a genuine gap between the bands persists, the situation is equivalent to the single-band case. The phase can be defined for each separate band and normalized by the number of states in the band. The results of the numerical simulations of this particular situation are shown in Fig. 1 along with $\tau(\kappa)$ obtained from our analytical Eq. (31). Here l_s was calculated numerically with the phase defined using the integral density of states normalized in such a way that the phase changes from zero to π when energy sweeps over a band from one fluctuation boundary to the other. In order to generate the plot we calculated both quantities τ and κ as functions of energy, and plotted them versus each other. One can see from this figure that the crossover between different asymptotes for both numerical and analytical calculations occurs in the same region. This proves the universal significance of our criterion (44) for SPS and justifies the suggested generalization of the crossover length l_s.

A new situation arises, however, when fluctuation states from adjacent bands overlap, and the spectrum does not have boundaries in former band gaps. On the one hand, it is clear that if the number of states in the former gap is small, SPS should not be expected on the basis of the general qualitative interpretation of the criterion (44). On the other hand, since there are no exact boundaries of the spectrum inside former gaps, one cannot define a phase suitable for determination of l_s. Though this situation requires special consideration, we can offer a conjecture that can be used to meaningfully define l_s in this case. Consider one of the original bands between two adjacent band gaps. In the presence of disorder there appear tails of the density of states within the gaps. When disorder is small, one can always distinguish between gap states originating from different bands (except for a small region where the tails from different bands overlap). Let $g_n(E)$ be the differential density of states related to the nth band. Then integral $N_{tot}=\int_{-\infty}^{\infty} g_n(E)dE$ gives the total number of states originating from the band. One can define a phase

$$\varphi_n(E)=\pi\int_{-\infty}^{E} g_n(E)dE/N_{tot},$$

with E obeying the inequality $E_{min}^{n-1}<E<E_{min}^{n}$, where E_{min}^{n} corresponds to the minimum of the *actual* differential density of states within the gap between the nth and $(n+1)$th bands. The phase defined according to this procedure does not assume unphysical values of π at the points where there are no spectrum boundaries, and we suggest that the parameter l_s defined through this phase according to

$$l_s^{-1}=\sin\left[\frac{\pi}{N_{tot}}\int_{-\infty}^{E} g_n(E)dE\right], \quad E_{min}^{n-1}<E<E_{min}^{n}, \quad (51)$$

can be used in order to formulate the criterion, Eq. (44). The suggested definition of l_s can be practically used for analytical estimates of the transition between different statistics, using, for example, a tight-binding approach to a multiband problem, where interactions only between adjacent bands are taken into account. However, more detailed discussion of this issue requires a separate paper.

V. PROPERTIES OF THE TRANSITION REGION BETWEEN SPS AND NON-SPS STATES

In this section we discuss properties of the transition region between SPS and non-SPS states. In spite of the mentioned peculiarities of the Lloyd model, our calculations provide a sound qualitative explanation for numerical results of Ref. 32, confirming once again that we correctly describe the qualitative nature of the transition between SPS and non-SPS statistics. In the model considered, the phase φ and LE γ can be conveniently presented in the form[4]

$$\sin\varphi=\frac{\sqrt{s}}{\sqrt{U_0^2+s}}, \quad (52)$$

$$\gamma\simeq\Gamma_U/\sqrt{s}, \quad (53)$$

where s is given as

$$s=\frac{1}{2}(4+\Gamma_U^2-U_0^2)+\frac{1}{2}\sqrt{(4+\Gamma_U^2-U_0^2)^2+4U_0^2\Gamma_U^2} \quad (54)$$

and we assume again that $\gamma\ll 1$. The relation between γ and φ is determined by the parameter s, which in its turn depends upon the energy E. Let us recall that the energy enters into our equations through parameters U_0 and Γ_U defined in Eqs. (8) and (9). For energies within conduction bands, the LE is of the order of Γ_U though l_s is of the order of 1 [see Eq. (48)]. Thus, SPS holds as long as disorder is small, $\Gamma_U\ll 1$, in accordance with the previous results.[2,6,29,30] From Eqs. (52), (53), and (54) it is clear that the relation between l_{loc} and l_s changes with the energy approaching an initial spectral boundary. In the limit of small disorder $l_{loc}=l_s$ exactly at the boundary. Therefore, one should expect the strongest violation of SPS for states which arise due to disorder in the originally forbidden regions. For the AM this corresponds to energies $|E|>2$, and for KPM's these are energies from band gaps of the original spectrum. For energies lying far away from the boundaries one can obtain the following approximate expressions for LE and the phase φ:

$$\sin\varphi\sim\frac{\Gamma_U}{\sqrt{U_0^2-4}}\ll 1, \quad (55)$$

$$\gamma\sim\frac{\sqrt{U_0^2-4}}{U_0}. \quad (56)$$

It is evident that in this case $l_{loc}\ll l_s$, and the variance behaves according to Eq. (46). The states disobeying SPS, however, are more important for KPM's than for the AM. The reason for this is that the LE in the AM becomes of the order of 1 not very far from the boundary, moving the system out of any scaling regime. In KPM's γ can remain small enough throughout entire band gaps for sufficiently high energies $k\gg V_0$, and the violation of SPS in this case occurs when the localization length is still of a macroscopic scale.

Equations (39) and (46) explain a nonmonotonic behavior of $\sigma(E)$ observed numerically in Ref. 32. When the energy moves towards a band edge, the LE grows and σ grows along with γ. When, however, γ becomes equal to l_s^{-1}, the variance σ^2 starts decreasing towards the value $\sim\pi l_s^{-1}/N$. The maximum of σ, therefore, corresponds to the energy where $\gamma\simeq l_s^{-1}$, i.e., the boundary of the original spectrum.

We can now also estimate the width of the transition region between SPS and non-SPS states, which was found in numerical simulations to be surprisingly small. The transition between the two groups of states occurs when $\sigma(E)$ passes through its maximum, and the width of the transition region is related to the sharpness of the maximum. In view of the preceding discussion, the latter is determined by

the region of energies over which $\gamma(\Gamma_U)$ changes its behavior. The extent of this region can be estimated from the condition

$$|4-U_0^2| \sim 2U_0\Gamma_U. \tag{57}$$

In the AM it leads to $\delta E \sim \Gamma$, and in KPM's one has

$$\delta k \sim \frac{\Gamma}{ak_n\delta_n} \sim \frac{\Gamma}{V_0}\frac{\Delta_n}{\delta_n}, \tag{58}$$

where Δ_n is the width of the nth band gap, k_n represents the nonresonant boundary of the nth band, and the parameter δ_n is defined as

$$\delta_n = 1 + \frac{4V_0}{ak_n^2}. \tag{59}$$

In both cases the width of the transition region is determined by the degree of disorder in the system and is small when disorder is small. In the Kronig-Penney situation, however, Eq. (58) indicates a special sharpness of the transition in the case of high-energy bands, when the parameter Δ_n is also small.

When disorder increases, the AM and KPM's behave differently. Monte Carlo results[32] show that in periodic systems, an increase of disorder leads to a restoration of SPS for almost the entire spectrum of the system. We are now able to explain this behavior and to provide an estimate for the critical disorder. It is clear that the parameter l_s^{-1} reaches a minimum at the energy in the center of a band gap. This minimum value can be estimated from Eq. (56) as

$$l_{s\,\min}^{-1} \sim \frac{\Gamma}{k_n}\sqrt{\delta_n}, \tag{60}$$

where k_n represents the nonresonant boundary of the nth band, and the parameter δ_n is defined by Eq. (59). At the same energy where l_s^{-1} is minimal, the LE assumes its maximum value:

$$\gamma_{\max} \sim \frac{V_0}{k_n\sqrt{\delta_n}}. \tag{61}$$

$l_{s\,\min}^{-1}$ increases with disorder, while $\gamma_{\max}$ does not change, and at $\Gamma \simeq V_0/\delta_n$ two length scales are of the same order, $l_{s\,\min} \simeq \gamma_{\max}^{-1}$. At this instant for the states outside of the immediate vicinity of the center of the band gap, $l_s \gg \gamma^{-1}$ and SPS is restored. Thus, we can identify $\Gamma_{cr}^n = V_0/\delta_n$ as a critical disorder for the nth band gap. For the states right in the center of the gaps, however, $l_s \sim \gamma^{-1}$ no matter what disorder is, and these states do not obey SPS. Therefore, the complete restoration of SPS for the entire band gap does not occur in this model, but the width of the non-SPS region decreases with increase of disorder. This can be seen from Fig. 2, where we present the LE and its variance obtained from our analytical results, Eq. (31), for different degrees of disorder. In the case of a strong potential V_0 and high energy, $a^{-1} \ll V_0 \ll k_n$, one has $\delta_n \simeq 1$, and the critical disorder is just

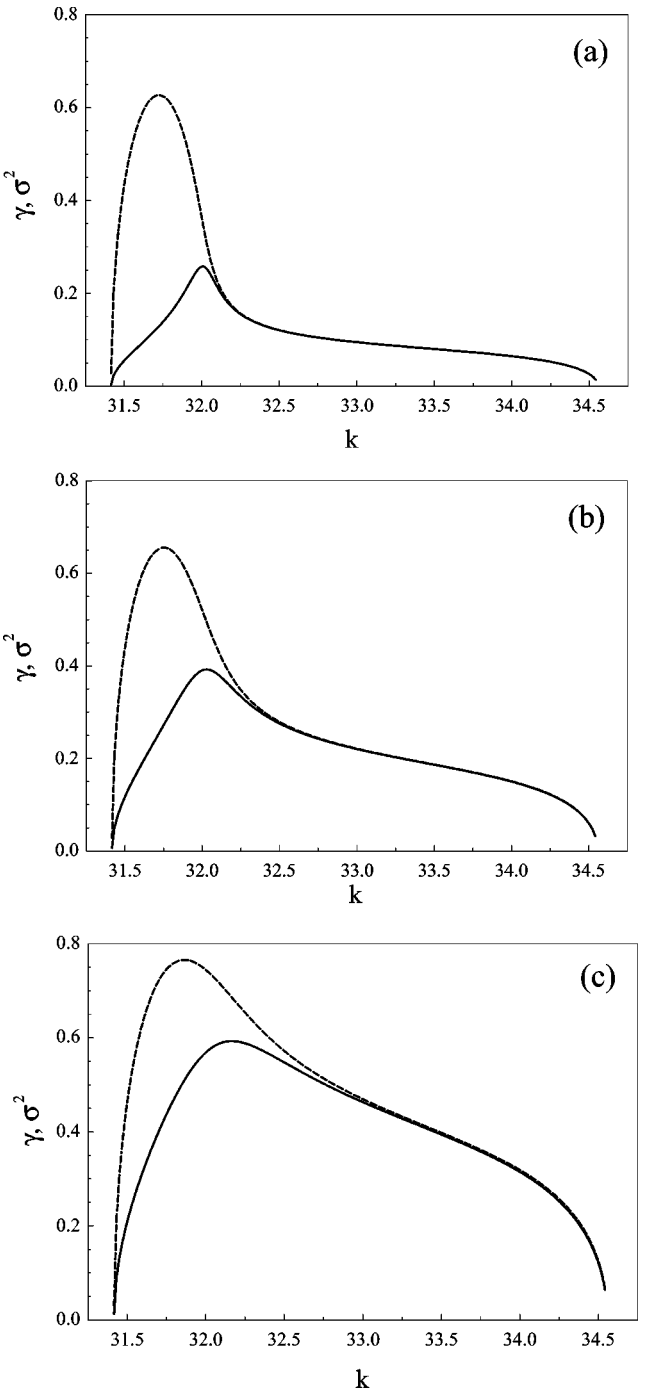

FIG. 2. The variance of the LE multiplied by the length of the system L (solid line) and LE itself (dashed line) for KPM's are posted as functions of the energy parameter k for different degrees of disorder. The plot extends over the region $10\pi < ka < 11\pi$, so one gap and one conductance band are represented. The presence of the maximum in the vicinity of the band boundary ($ka \sim 10.35\pi$) signals the violation of SPS for states from a band gap. The absence of a similar maximum at the second boundary is due to its resonance nature. The degrees of the disorder, Γ, are 3 (a), 7 (b), and 15 (c). It is clearly seen that with the increase of disorder, the number of states disobeying SPS decreases.

equal to V_0. When a potential is weak, $V_0 a \ll 1$, there exists a medium-energy regime, when $\sqrt{V_0/a} \gg k_n \gg V_0$. In this case

$$\gamma_{\max} \sim \sqrt{V_0 a} \ll 1,$$

$$l_{s\,\min}^{-1} \sim \frac{\Gamma}{k_n^2}\sqrt{\frac{V_0}{a}},$$

$$\Gamma_{cr}^n \simeq a k_n^2.$$

It is interesting to note that in this case Γ_{cr}^n increases with an increase of k_n, even though the widths of the gaps decrease.

The calculations presented above referred to the nonresonant band boundaries. At the resonant points $ka = \pi n$, both the localization length l_{loc} and the crossover length l_s diverge, while the variance of the LE vanishes. Although the resonances are not stable with respect to a violation of the periodic arrangements of the δ potentials, they occur in some other models as well. We already mentioned models with correlated disorder[38] and random superlattices.[25,32] The latter has an experimental significance with applications to propagation of classical waves. Therefore, it is interesting to consider the behavior of the critical parameter κ in the vicinity of the resonances. Although both l_{loc} and l_s diverge at the resonances, their ratio κ remains finite and takes on the following values:

$$\kappa = \begin{cases} \dfrac{V_0}{\Gamma_0} + \sqrt{1 + \dfrac{V_0}{\Gamma_0}}, & ka < \pi n, \\ -\dfrac{V_0}{\Gamma_0} + \sqrt{1 + \dfrac{V_0}{\Gamma_0}}, & ka > \pi n. \end{cases} \tag{62}$$

One can see from Eq. (62) that κ experiences a discontinuity at resonance points: its value decreases by $2V_0/\Gamma_0$ once a point is crossed. In the case of small disorder, when $V_0/\Gamma_0 \gg 1$, this is a dramatic jump, such that $\kappa \gg 1$ at the band side of the resonant boundary and $\kappa \ll 1$ at the gap side. It is obvious, therefore, that the scaling properties of the system also change discontinuously at the resonance from SPS behavior at the band side to the scaling with two parameters at the gap side.

VI. CONCLUSION

In this paper we studied statistical properties of the Lyapunov exponent in the one-dimensional Anderson model with the Cauchy distribution of site energies. The model can also be interpreted as the Kronig-Penney model with periodically positioned δ potentials with random strengths. The main objective of the study was to find an exact solution for the thermodynamical limit of the variance of the LE and to establish an exact criterion for the existence of single parameter scaling. It is important to emphasize that in contrast with all previous calculations of the variance, we did not use the phase randomization hypothesis. This allowed us to reject the generally accepted assumption that it is the length over which the phase of reflection and transmission coefficients become uniformly distributed that sets the condition for the existence of SPS.

We found a new length scale l_s, which is responsible for the scaling properties of the conductivity in the system: SPS exists as long as the localization length l_{loc} exceeds l_s. The length l_s, however, differs from the phase randomization length and presents, therefore, a new significant scaling parameter. The parameter l_s is microscopic for states close to the center of the original conduction bands of the system and does not impose, therefore, any additional restrictions for the existence of SPS excepting the regular requirement for the localization length to be of a macroscopic dimension. However, for the states at the edge of the bands, l_s grows to a macroscopic size and the condition $l_{loc} = l_s$ actually establishes a boundary between the states with and without the SPS statistics. As soon as l_s becomes much larger than all microscopic lengths, this scale becomes significant. In this limit it can be expressed in terms of the number of states, $N_{def}(E)$, which arise at the tails of the initial bands due to rare fluctuation configurations: $l_s^{-1} = N_{def}(E)$. It then can be given a natural physical interpretation as an average distance between such defects.

The change of the scaling behavior occurs when the energy crosses over a boundary of a former gap. In the case of regular boundaries, the change occurs gradually with the critical parameter κ being of the order of unity right at the boundary. The Kronig-Penney version of our model, besides regular boundaries, has so-called resonant boundaries, where both the LE and l_s^{-1} vanish. We found that at the resonance boundaries the parameter κ undergoes a sudden jump from very large values $\kappa \gg 1$ at the band side of the boundary to very small values $\kappa \ll 1$ at the gap side. This means that the change of the scaling behavior at the resonant energies also occurs discontinuously: the system obeys SPS when the boundary is approached from the conduction band and then demonstrates two-parameter scaling if the boundary is approached from the gap.

We carried out numerical simulations of the Kronig-Penney-like model with a different configuration of the potential and different statistics. The comparison between numerical and analytical results clearly indicates that significance of the length scale l_s defined in terms of the integral density of states persists beyond the Lloyd model and that the new criterion for SPS established in the present paper has a universal nature.

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