

REVIEWS OF TOPICAL PROBLEMS

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PT-symmetry in optics

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Contents

Abstract. We review quantum mechanical and optical pseudo-Hermitian systems with an emphasis on PT-symmetric systems important for optics and electrodynamics. One of the most interesting and much discussed consequences of PT symmetry is a phase transition under which the system eigenvalues lose their PT symmetry. We show that although this phase transition is difficult to realize experimentally, a similar transition can be observed in quasi-PT-symmetric systems. Other effects predicted for PT-symmetric systems are not specific for these systems and can be observed in ordinary fully passive systems.

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1. Introduction

Interest in the optics of artificial heterogeneous media rekindled in the last decade. These media have many unique properties that are absent in homogeneous natural materials, including artificial magnetism at optical frequencies [1–4], negative refraction [5, 6], and strong spatial dispersion [7, 8]. These new properties, related to the resonance nature of the interaction of light with materials, are observed in plasma systems [9], photonic crystals [10, 11], random lasers [12], and other systems. Unfortunately, the resonance interaction does not always lead to the enhancement of useful properties only. Quite often, Joule losses strongly increase in this case, restricting applications of such media [13–15]. To solve this problem, it was proposed in [16–24] to add active (amplifying) components to a heterogeneous system. As a result, a wave is amplified in some regions of the system and is attenuated in others. Heuristically it seems that losses can be compensated by ensuring equal volumes and equal imaginary parts of the amplifying and dissipative components. It turns out that under these conditions, a transition can occur from solutions that experience neither amplification nor attenuation to solutions that, despite the apparent compensation of losses, are either amplified or attenuated [25–27]. The solution type changes at the transition point. An important example of such systems, which allow obtaining analytically rigorous results, is parity–time (PT) symmetric systems. The PT symmetry in optical systems amounts to the condition $\varepsilon(x,z) = \varepsilon^*(-x,-z)$, where $\varepsilon(x,z)$ is the permittivity of the medium.

In this review, we consider optical phenomena observed in a new type of optical heterogeneous media with PT symmetry.

Interest in such problems initially appeared in quantum mechanical studies [28, 29] of Hamiltonians with a complex potential satisfying the condition $V(\mathbf{r}) = V^*(-\mathbf{r})$. In Section 2, we therefore consider the quantum mechanics of PT-symmetric and pseudo-Hermitian systems. We discuss the quantum mechanical properties of spatial inversion and time reversal operators $\hat{P}$ and $\hat{T}$ in detail, analyze the necessary and sufficient conditions for the existence of real eigenvalues of a pseudo-Hermitian (in particular, PT-symmetric) Hamiltonian, and consider a phase transition related to the PT symmetry breaking for the eigenstates of such a quantum system. The eigenvalues of such a Hamiltonian turn out to be real. At first glance, it would seem natural to extend the existing quantum mechanics by including systems with pseudo-Hermitian Hamiltonians into consideration. But it turned out that in order to consistently develop quantum mechanics with pseudo-Hermitian Hamiltonians, it is necessary to redefine the scalar product of wave functions, which makes the combination of this new quantum mechanics with the traditional one impossible.

In Section 3, we discuss PT-symmetric optical systems and their possible practical realizations. We show that exact conditions for the PT symmetry of the system can be satisfied only for a discrete set of real frequencies.

In Section 4, two-dimensional PT-symmetric systems are considered. An analogy is pointed out between the Helmholtz equation for a two-dimensional optical system and the Schrödinger equation for a one-dimensional quantum mechanical system. Various phenomena are analyzed that can be observed in PT-symmetric systems. In particular, conditions of the refraction asymmetry in optical systems are discussed. We consider phase transitions in two-dimensional PT-symmetric systems and in PT-nonsymmetric optical systems that can be mapped onto PT-symmetric ones by coordinate transformation. Their behavior is analyzed.

In Section 5, the properties of one-dimensional PT-symmetric optical systems are discussed. We analyze conditions for the occurrence of a phase transition and the possibility of developing a device based on PT-symmetric systems that would simultaneously operate as a laser and an ideal absorber.

In Section 6, we critically review the discussion about the reciprocity of optical PT-symmetric systems. We consider conditions that an optical system should satisfy to operate as an optical diode.

In conclusion (Section 7), we show that although a phase transition with PT-symmetry breaking cannot be experimentally realized by changing frequency and is difficult to realize by changing other parameters, a phenomenon similar to a phase transition can be observed in most PT-invariant systems. All the other phenomena predicted in the literature for PT-symmetric media are not specific to PT-invariant systems and can also be observed in regular passive systems.

2. PT symmetry. Basic concepts and definitions

In 1998, Bender and Boettcher [28] showed that quantum systems with a non-Hermitian Hamiltonian can have a set of eigenstates with real eigenvalues (a real spectrum). In other words, they found that the Hermiticity of the Hamiltonian is not a necessary condition for the realness of its eigenvalues, and new quantum mechanics can be constructed based on such Hamiltonians [28, 30, 31].

The initial point of such a construction is the following fact. In the case of real eigenvalues of a non-Hermitian Hamiltonian, the modulus of the wave function for the eigenstates of the system is conserved in time even in regions with a complex potential. Indeed,

$$\hat{H}\psi_k = E_k\psi_k \tag{1}$$

for any eigenstate of the Hamiltonian. Substituting (1) in the time-dependent Schrödinger equation, we obtain

$$\mathrm{i}\,\frac{\partial\psi_k}{\partial t} = E_k\psi_k\,. \tag{2}$$

Obviously, for any real E_k, the modulus of ψ_k is conserved in time. However, the eigenstates of such a Hamiltonian are not orthogonal, and constructing self-consistent quantum mechanics based on such Hamiltonians requires redefining the scalar product and norm [32] (see Appendix 1).

2.1 Parity and time reversal operators

The historically first pseudo-Hermitian Hamiltonian with a real spectrum was a PT-symmetric Hamiltonian [28]. The PT-symmetry of the Hamiltonian means that it commutes with the time reversal operator $\hat{T}$ and the parity operator $\hat{P}$:

$$\hat{P}\hat{T}\hat{H} = \hat{H}\hat{P}\hat{T}\,. \tag{3}$$

The action of the parity operator $\hat{P}$ amounts to the change of sign of all coordinates ($x \to -x$, $y \to -y$, $z \to -z$) [33]. As a result, three unit vectors pass from a right to a left coordinate system, polar vectors change their direction to the opposite ($\mathbf{r} \to -\mathbf{r}$, $\mathbf{p} \to -\mathbf{p}$, $\mathbf{E} \to -\mathbf{E}$), while axial vectors do not change ($\mathbf{H} \to \mathbf{H}$). Here, $\mathbf{r}$ is the spatial coordinate, $\mathbf{p}$ is the momentum, and $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic fields.

The mean value of a physical quantity operator in quantum mechanics corresponds to the classical value of this quantity. Because the classical momentum and coordinate change their signs under spatial inversion, this means that $\langle\mathbf{p}\rangle$ and $\langle\mathbf{r}\rangle$ should also change their signs. Therefore, the momentum and coordinate operators are transformed under spatial inversion by the rule

$$\hat{P}^+\hat{\mathbf{r}}\hat{P} = -\hat{\mathbf{r}}\,, \tag{4a}$$

$$\hat{P}^+\hat{\mathbf{p}}\hat{P} = -\hat{\mathbf{p}}\,, \tag{4b}$$

where $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ are the coordinate and momentum operators. According to this, the angular momentum operator $\hat{\mathbf{j}}$ remains unchanged under spatial inversion:

$$\hat{P}^{+}\hat{\mathbf{j}}\hat{P} = \hat{\mathbf{j}}\,. \tag{4c}$$

In addition, the normalization of the wave function should be preserved under spatial inversion, and therefore the parity operator is unitary, $\hat{P}^{+}\hat{P} = \hat{1}$.

According to the Wigner theorem [343, 35], symmetry operators can be either linear and unitary or antilinear and antiunitary. According to the definition, a linear operator does not change c-number factors in equations:

$$\hat{Q}_{\mathrm{L}} c\psi(\mathbf{r}, t) = c\hat{Q}_{\mathrm{L}}\,\psi(\mathbf{r}, t)\,, \tag{5a}$$

while an antilinear operator leads to their complex conjugation:

$$\hat{Q}_{\mathrm{AL}} c\,\psi(\mathbf{r}, t) = c^{*}\hat{Q}_{\mathrm{AL}}\,\psi(\mathbf{r}, t)\,. \tag{5b}$$

Hence, to find whether the operator $\hat{P}$ is linear or antilinear, it is necessary to determine its action on the imaginary unit: $\hat{P}^{+}\mathrm{i}\hat{1}\hat{P}$. Because the canonical commutation relations

$$[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = \mathrm{i}\hat{1}\hbar \tag{6}$$

should remain invariant under spatial inversion [36], we obtain

$$\begin{aligned}\hat{P}^{+}\mathrm{i}\hat{1}\hat{P}\hbar &= \hat{P}^{+}[\hat{\mathbf{r}}, \hat{\mathbf{p}}]\hat{P} = \hat{P}^{+}\hat{\mathbf{r}}\hat{\mathbf{p}}\hat{P} - \hat{P}^{+}\hat{\mathbf{p}}\hat{\mathbf{r}}\hat{P}\\ &= \hat{P}^{+}\hat{\mathbf{r}}\hat{P}\hat{P}^{+}\hat{\mathbf{p}}\hat{P} - \hat{P}^{+}\hat{\mathbf{p}}\hat{P}\hat{P}^{+}\hat{\mathbf{r}}\hat{P} = \hat{\mathbf{r}}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{r}} = [\hat{\mathbf{r}}, \hat{\mathbf{p}}] = \mathrm{i}\hat{1}\hbar\,,\end{aligned} \tag{7}$$

where we took into account that $\hat{P}^{+}\hat{P} = \hat{1}$. Therefore, $\hat{P}^{+}\mathrm{i}\hat{1}\hat{P} = \mathrm{i}\hat{1}$ and $\hat{P}$ is a linear unitary operator [34].[1]

Because the double application of the parity operation returns the system to the initial state, the wave functions $\psi(\mathbf{r}, t)$ and $\hat{P}^{2}\psi(\mathbf{r}, t)$ can differ only by a phase factor: $\hat{P}^{2}\psi(\mathbf{r}, t) = \exp(\mathrm{i}\phi)\psi(\mathbf{r}, t)$. For the parity of the wave function to be an observable, the spatial inversion operator should be Hermitian. The only phase factor for which the spatial inversion operator is Hermitian, $\hat{P}^{+} = \hat{P}$, is unity: $\hat{P}^{2}\psi(\mathbf{r}, t) = \psi(\mathbf{r}, t)$. Assuming that the wave function is scalar and taking into account that $\hat{P}$ is a linear unitary operator [33], we obtain that

$$\hat{P}\psi(\mathbf{r}, t) = \psi(-\mathbf{r}, t)\,. \tag{8}$$

A unitary transformation of any product of momentum and coordinate operators reduces to the same product of transformed operators:

$$\begin{aligned}&\hat{P}^{+}\hat{\mathbf{r}}\hat{\mathbf{p}}\hat{P} = (\hat{P}^{+}\hat{\mathbf{r}}\hat{P})(\hat{P}^{+}\hat{\mathbf{p}}\hat{P})\,,\\ &\hat{P}^{+}\hat{\mathbf{r}}^{2}\hat{\mathbf{p}}\hat{P} = (\hat{P}^{+}\hat{\mathbf{r}}\hat{P})(\hat{P}^{+}\hat{\mathbf{r}}\hat{P})(\hat{P}^{+}\hat{\mathbf{p}}\hat{P})\,, \quad \ldots\,.\end{aligned} \tag{9}$$

If the Hamiltonian of a system can be represented in the form of a polynomial in momentum and coordinate operators, then

$$\hat{P}^{+}\hat{H}(\hat{\mathbf{p}}, \hat{\mathbf{r}}, t)\hat{P} = \hat{H}(\hat{P}^{+}\hat{\mathbf{p}}\hat{P}, \hat{P}^{+}\hat{\mathbf{r}}\hat{P}, t) = \hat{H}(-\hat{\mathbf{p}}, -\hat{\mathbf{r}}, t)\,. \tag{10}$$

[1] The operator $\hat{U}$ is called unitary if $\langle\hat{U}\psi|\hat{U}\varphi\rangle = \langle\psi|\varphi\rangle$, and antiunitary if $\langle\hat{U}\psi|\hat{U}\varphi\rangle = \langle\varphi|\psi\rangle$. In both cases, $\hat{U}$ preserves the norm of the wave function, because $\hat{U}^{+}\hat{U} = \hat{1}$ [37].

A system is $\hat{P}$-invariant if its Hamiltonian does not change after the inversion of coordinates, i.e.,

$$\hat{H}(\hat{\mathbf{p}}, \hat{\mathbf{r}}, t) = \hat{H}(-\hat{\mathbf{p}}, -\hat{\mathbf{r}}, t)\,. \tag{11}$$

The action of the time reversal operator $\hat{T}$ means the change $t \to -t$ in all equations and time dependences of physical quantities [33]. As a result, all physical quantities linearly dependent on the time derivative change their sign under time reversal ($\mathbf{p} \to -\mathbf{p}$, $\mathbf{j} \to -\mathbf{j}$), whereas the time-independent physical quantities do not change ($\mathbf{r} \to \mathbf{r}$).

Acting as in the derivation of relations (4a), (4b), and (4c), we obtain the rules for transformations of operators under time reversal:

$$\hat{T}^{+}\hat{\mathbf{r}}\hat{T} = \hat{\mathbf{r}}\,, \tag{12a}$$

$$\hat{T}^{+}\hat{\mathbf{p}}\hat{T} = -\hat{\mathbf{p}}\,, \tag{12b}$$

$$\hat{T}^{+}\hat{\mathbf{j}}\hat{T} = -\hat{\mathbf{j}}\,. \tag{12c}$$

Time reversal preserves the normalization of the wave function. To find whether the time reversal operator is linear or antilinear, it is necessary to determine its action on the imaginary unit: $\hat{T}^{+}\mathrm{i}\hat{1}\hat{T}$. Similarly to (6) and (7), we obtain

$$\begin{aligned}\hat{T}^{+}\mathrm{i}\hat{1}\hat{T}\hbar &= \hat{T}^{+}[\hat{\mathbf{r}}, \hat{\mathbf{p}}]\hat{T} = \hat{T}^{+}\hat{\mathbf{r}}\hat{\mathbf{p}}\hat{T} - \hat{T}^{+}\hat{\mathbf{p}}\hat{\mathbf{r}}\hat{T}\\ &= \hat{T}^{+}\hat{\mathbf{r}}\hat{T}\hat{T}^{+}\hat{\mathbf{p}}\hat{T} - \hat{T}^{+}\hat{\mathbf{p}}\hat{T}\hat{T}^{+}\hat{\mathbf{r}}\hat{T} = -\hat{\mathbf{r}}\hat{\mathbf{p}} + \hat{\mathbf{p}}\hat{\mathbf{r}} = -[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = -\mathrm{i}\hat{1}\hbar\,.\end{aligned} \tag{13}$$

This means that $\hat{T}$ is an antilinear and antiunitary operator [34] (see footnote 1).

Because the double action of the time reversal operation returns the system to the initial state, the wave functions $\psi(\mathbf{r}, t)$ and $\hat{T}^{2}\psi(\mathbf{r}, t)$ can differ only by a phase factor: $\hat{T}^{2}\psi(\mathbf{r}, t) = \exp(\mathrm{i}\phi)\psi(\mathbf{r}, t)$. To determine the phase factor, we apply the operator $\hat{T}^{3}$ to the wave function $\psi(\mathbf{r}, t)$:

$$\begin{aligned}\hat{T}^{3}\psi(\mathbf{r}, t) &= \hat{T}\big(\hat{T}^{2}\psi(\mathbf{r}, t)\big) = \hat{T}\big(\exp(\mathrm{i}\phi)\,\psi(\mathbf{r}, t)\big)\\ &= \exp(-\mathrm{i}\phi)\hat{T}\psi(\mathbf{r}, t) = \hat{T}^{2}\big(\hat{T}\psi(\mathbf{r}, t)\big)\,.\end{aligned} \tag{14}$$

Hence, the function $\hat{T}\psi(\mathbf{r}, t)$ acquires the phase factor $\exp(-\mathrm{i}\phi)$ under the action of $\hat{T}^{2}$, while the wave function $\psi(\mathbf{r}, t)$ acquires the phase factor $\exp(\mathrm{i}\phi)$. Because the function $\psi(\mathbf{r}, t) + \hat{T}\psi(\mathbf{r}, t)$ can change under the action of $\hat{T}^{2}$ only by a common phase factor, we have $\exp(-\mathrm{i}\phi) = \exp(\mathrm{i}\phi)$ and $\hat{T}^{2}\psi(\mathbf{r}, t) = \pm\psi(\mathbf{r}, t)$. Assuming that the wave function is a scalar and taking into account that $\hat{T}$ is an antilinear and antiunitary operator, we obtain the time reversal rule for the wave function [33]:

$$\hat{T}\psi(\mathbf{r}, t) = \psi^{*}(\mathbf{r}, -t)\,. \tag{15}$$

In particular, for a plane wave $\psi(x, p, t) = A\exp(-\mathrm{i}\omega t + \mathrm{i}kr)$, the action of the time reversal operator changes the propagation direction to the opposite one:

$$\hat{T}\big[A\exp(-\mathrm{i}\omega t + \mathrm{i}kr)\big] = A^{*}\exp(-\mathrm{i}\omega t - \mathrm{i}kr)\,. \tag{16}$$

Because the operator $\hat{T}$ is antiunitary, relation (9) also holds for it, i.e.,

$$\begin{aligned}&\hat{T}^{+}\hat{\mathbf{r}}\hat{\mathbf{p}}\hat{T} = (\hat{T}^{+}\hat{\mathbf{r}}\hat{T})(\hat{T}^{+}\hat{\mathbf{p}}\hat{T})\,,\\ &\hat{T}^{+}\hat{\mathbf{r}}^{2}\hat{\mathbf{p}}\hat{T} = (\hat{T}^{+}\hat{\mathbf{r}}\hat{T})(\hat{T}^{+}\hat{\mathbf{r}}\hat{T})(\hat{T}^{+}\hat{\mathbf{p}}\hat{T})\,.\end{aligned} \tag{17}$$

Similarly to the derivation of (10), we find

$$\hat{T}^{+}\hat{H}(\hat{\mathbf{p}},\hat{\mathbf{r}},t)\hat{T} = \hat{H}^{*}(\hat{T}^{+}\hat{\mathbf{p}}\hat{T},\hat{T}^{+}\hat{\mathbf{r}}\hat{T},t) = \hat{H}^{*}(-\hat{\mathbf{p}},\hat{\mathbf{r}},t)\,. \quad (18)$$

A system is $\hat{T}$-invariant if its Hamiltonian does not change under time reversal, i.e.,

$$\hat{H}(\hat{\mathbf{p}},\hat{\mathbf{r}},t) = \hat{H}^{*}(-\hat{\mathbf{p}},\hat{\mathbf{r}},t)\,. \quad (19)$$

By combining conditions (10) and (19), we obtain the transformation of the Hamiltonian under the simultaneous action of $\hat{P}$ and $\hat{T}$:

$$\hat{P}^{+}\hat{T}^{+}\hat{H}(\hat{\mathbf{p}},\hat{\mathbf{r}},t)\hat{P}\hat{T} = \hat{H}(\hat{P}^{+}\hat{T}^{+}\hat{\mathbf{p}}\hat{P}\hat{T},\hat{P}^{+}\hat{T}^{+}\hat{\mathbf{r}}\hat{P}\hat{T},t)$$
$$= \hat{H}^{*}(\hat{\mathbf{p}},-\hat{\mathbf{r}},-t)\,. \quad (20)$$

Therefore, a Hamiltonian is PT symmetric if

$$\hat{H}(\hat{\mathbf{p}},\hat{\mathbf{r}},t) = \hat{H}^{*}(\hat{\mathbf{p}},-\hat{\mathbf{r}},-t)\,. \quad (21)$$

Condition (21) can be represented in a form similar to condition (3), $\hat{P}\hat{T}\hat{H}(\mathbf{p},\mathbf{r},t) = \hat{H}(\mathbf{p},\mathbf{r},t)\hat{P}\hat{T}$. For Hamiltonians of the form

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r})\,, \quad (22)$$

where m is the mass and V is the potential energy of a particle, the PT-symmetry condition (21) reduces to the requirement that the real part of the potential be an even function of the coordinate and the imaginary part be an odd function:

$$V(\mathbf{r}) = V^{*}(-\mathbf{r})\,. \quad (23)$$

For a non-Hermitian PT-symmetric Hamiltonian $\hat{H}(\hat{\mathbf{p}},\hat{\mathbf{r}},t,\mu)$ depending on a parameter μ, the eigenvalues of the system can be both real and complex. If the real eigenvalue spectrum of the Hamiltonian changes to a complex spectrum upon varying μ,[2] we are dealing with a phase transition [28, 31].

2.2 Necessary and sufficient condition for real eigenvalues of a PT-symmetric Hamiltonian

For the eigenvalues $\{E_k\}$ of a PT-symmetric system to be real, it is necessary and sufficient that its eigensolutions $\{\psi_k\}$ be PT symmetric [28, 31]. Indeed, let ψ_k be a PT-symmetric eigensolution of the system with a Hamiltonian $\hat{H}$,

$$\psi_k = \hat{P}\hat{T}\psi_k\,, \quad (24)$$

with an eigenvalue E_k,

$$\hat{H}\psi_k = E_k\psi_k\,. \quad (25)$$

We successively apply the operator $\hat{P}\hat{T}$ to both parts of Eqn (25):

$$\hat{P}\hat{T}\hat{H}\psi_k = \hat{P}\hat{T}E_k\psi_k\,. \quad (26)$$

If the system Hamiltonian $\hat{H}$ satisfies PT-symmetry condition (3), (21), then, with the PT-symmetry of ψ_k in (24) taken into account, the left-hand side of (26) takes the form

$$\hat{P}\hat{T}\hat{H}\psi_k = \hat{H}\psi_k\,. \quad (27)$$

[2] μ can be the particle charge, the coupling constant, and so on.

Because the parity operator $\hat{P}$ is linear, and the time reversal operator $\hat{T}$ is antilinear, the right-hand side of (26) is transformed as

$$\hat{P}\hat{T}E_k\psi_k = E_k^{*}\psi_k\,. \quad (28)$$

Combining (27) and (28), we obtain

$$\hat{H}\psi_k = E_k^{*}\psi_k\,. \quad (29)$$

Equations (25) and (29) are compatible only for $E_k = E_k^{*}$. Hence, the PT symmetry of a solution ψ_k is a sufficient condition for the realness of the eigenvalue E_k of a PT-symmetric Hamiltonian $\hat{H}$.

We now prove that the PT symmetry of the solution ψ_k is also a necessary condition for the realness of the eigenvalues of a PT-symmetric Hamiltonian with a nondegenerate spectrum. We assume that the eigenvalues of the system are real, $E_k \in \mathbb{R}$, and the eigensolutions ψ_k are not PT symmetric:

$$\hat{P}\hat{T}\psi_k = \psi'\,, \quad (30)$$

where ψ' is an unknown wave function. We apply the operator $\hat{P}\hat{T}$ to both sides of the stationary Schrödinger equation (25) for ψ_k:

$$\hat{P}\hat{T}\hat{H}\psi_k = \hat{P}\hat{T}E_k\psi_k\,. \quad (31)$$

In view of condition (3) of the PT symmetry of the Hamiltonian and the realness of its eigenvalues E_k, we can write

$$\hat{H}(\hat{P}\hat{T}\psi_k) = E_k(\hat{P}\hat{T}\psi_k)\,, \quad (32)$$

whence, using Eqn (30), we obtain that

$$\hat{H}\psi' = E_k\psi'\,. \quad (33)$$

Therefore, ψ' is an eigenfunction of the Hamiltonian $\hat{H}$ with the eigenvalue E_k. If the spectrum of $\hat{H}$ is nondegenerate, then $\psi' = \psi_k$, and it follows from (30) that ψ_k is an eigenfunction of the operator $\hat{P}\hat{T}$:

$$\hat{P}\hat{T}\psi_k = \psi' = \psi_k\,. \quad (34)$$

The case of a PT-symmetric Hamiltonian with a degenerate spectrum was considered in review [31], where it was shown that the PT symmetry of the eigensolutions of the Hamiltonian is also a necessary and sufficient condition for the realness of the eigenvalues. In other words, as long as the eigenvalues E_k of the Hamiltonian are real, its eigensolutions are PT symmetric. If the eigenvalues are complex, then the eigenfunctions of the Hamiltonian are not eigenfunctions of the operator $\hat{P}\hat{T}$.

We note that the real eigenvalues are inherent in the spectra of a broader class of *pseudo-Hermitian* systems (see Appendix 2). However, in optics we are mainly dealing with PT-symmetric systems, and we discuss their properties below.

2.3 Phase transition in PT-symmetric systems

As shown in Section 2.2, when the eigenvalues are real, the system is in a PT-symmetric phase, whereas for complex eigenvalues, the system is in a PT-nonsymmetric phase [31]. If real eigenvalues change to complex ones upon varying some parameter μ of the Hamiltonian, we are dealing with a second-order phase transition [38] related to a spontaneous PT-symmetry breaking for eigensolutions.

A change in the symmetry in passing through a phase transition point is usually quantitatively described by introducing an order parameter η, which is zero in the symmetric phase and nonzero in the nonsymmetric phase [38]. In the case of PT-symmetric systems, we can take this parameter in the form

$$\eta = \sum_k |\psi_k^* \psi_k - \hat{P}\hat{T}\psi_k^* \psi_k| \,, \tag{35}$$

where the summation is performed over all eigenstates of the system. The order parameter η introduced in this way is zero in the symmetric phase and increases with increasing nonsymmetry of the eigensolutions. Another possible quantity meeting the formal requirements imposed on the order parameter is the sum of moduli of the imaginary parts of the Hamiltonian eigenvalues, $\eta = \sum_k |\mathrm{Im}\, E_k|$. In the mean-field Ginzburg–Landau theory, the order parameter is used for the phenomenological description of the phase transition [38]. However, in the case of PT-symmetric systems, it is usually simpler to find the eigensolutions ψ_k of the system and determine the phase transition point from them than to use the order-parameter formalism.

The symmetry operator in conventional second-order phase transitions is linear. In this case, the phase transition and spontaneous symmetry breaking of the solution are possible only when degeneracy appears [39]. In the one-dimensional case, all energy levels of the discrete spectrum are nondegenerate, and phase transitions in a system with a linear symmetry transformation and a discrete spectrum are impossible.

The PT-symmetric systems in electrodynamics differ from systems in statistical physics that are invariant under linear symmetry transformations in that because of the antilinearity of the $\hat{P}\hat{T}$ operator, the phase transition can be observed in one-dimensional PT-symmetric systems with a discrete spectrum. Indeed, if a linear symmetry operator commutes with the Hamiltonian, then their nondegenerate eigensolutions always coincide. However, if an antilinear symmetry operator commutes with the Hamiltonian, their eigensolutions coincide only if the eigenvalues are real (see Section 2.2).

3. PT-symmetric optical systems

3.1 PT-symmetry concept for optical systems

In Section 2, we considered the main properties of PT-symmetric quantum mechanical systems and discussed the possibility of observing a second-order phase transition in such systems. In [25, 26], the quantum mechanical PT-symmetry concept was extended to optics.

The PT-symmetry concept in optics can be introduced in the following way. In the two- and one-dimensional cases, Maxwell's equations reduce to the scalar Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 \varepsilon(x,z)\right) E(x,z) = 0 \,, \tag{36a}$$

which formally coincides with the stationary Schrödinger equation [41]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \psi_k(x,z) - \frac{2m\big(V(x,z) - E_k\big)}{\hbar^2} \, \psi_k(x,z) = 0 \,, \tag{36b}$$

under the formal substitutions $V(x,z) - E_k \to (\omega/c)^2 \varepsilon(x,z)$, $\psi_k(x,z) \to E(x,z)$, and $-\hbar^2/(2m) \to 1$. The PT-symmetry condition for a quantum mechanical system described by Eqn (36b) is reduced to the requirement $V(x,z) = V^*(-x,-z)$ imposed on the potential energy. Therefore, by analogy between the potential energy in quantum mechanics and the permittivity in optics, the PT-symmetry condition for the optical system is defined as the condition imposed on the permittivity of the medium:

$$\mathrm{Re}\, \varepsilon(\omega, x, z) = \mathrm{Re}\, \varepsilon(\omega, -x, -z) \,, \tag{37a}$$

$$\mathrm{Im}\, \varepsilon(\omega, x, z) = -\mathrm{Im}\, \varepsilon(\omega, -x, -z) \,. \tag{37b}$$

We note that the stationary Schrödinger equation does not include the time dependence, and therefore the time reversal operation $\hat{T}$ is equivalent to the complex conjugation $\hat{K}$. Below, the T-symmetry of the system in optics means its K-symmetry, and we continue to call systems satisfying condition (37) PT-symmetric systems.

3.2 Restrictions imposed by the causality principle. Kramers–Kronig relations and the possibility of PT-symmetric optical systems existing in a finite frequency range

The permittivity $\varepsilon(\omega, \mathbf{r})$, like any response function, must be causal (an analytic function without poles in the upper half-plane of the complex frequency plane) [42]. It follows from this requirement that the real and imaginary parts of the permittivity are connected by the Kramers–Kronig relations [42]

$$\mathrm{Re}\, \varepsilon(\omega, \mathbf{r}) = \varepsilon_0 + \frac{1}{\pi} \, \mathrm{v.p.} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, \varepsilon(\omega', \mathbf{r})}{\omega' - \omega} \, \mathrm{d}\omega' \,, \tag{38a}$$

$$\mathrm{Im}\, \varepsilon(\omega, \mathbf{r}) = -\frac{1}{\pi} \, \mathrm{v.p.} \int_{-\infty}^{\infty} \frac{\mathrm{Re}\, \varepsilon(\omega', \mathbf{r}) - \varepsilon_0}{\omega' - \omega} \, \mathrm{d}\omega' \,, \tag{38b}$$

where ε_0 is the vacuum permittivity and integrals are taken in the sense of principal value (valeur principale, v.p.). Conditions (37) show that any nontrivial PT-symmetric optical system consists of amplifying media with $\varepsilon_{\mathrm{gain}}(\omega)$ for which $\mathrm{Im}\, \varepsilon_{\mathrm{gain}} < 0$ and absorbing media with $\varepsilon_{\mathrm{pass}}(\omega)$ for which $\mathrm{Im}\, \varepsilon_{\mathrm{pass}} > 0$. The nonzero imaginary part of the permittivity means that the permittivity $\varepsilon(\omega, \mathbf{r})$ has a frequency dispersion. Hence, condition (37) of the PT-symmetry of the system can be satisfied only for a discrete frequency set [43]. Indeed, if PT-symmetry condition (37b) is satisfied for the imaginary part $\mathrm{Im}\, \varepsilon(\omega)$ of the permittivity for any real frequencies, then

$$\begin{aligned} \mathrm{Re}\, \varepsilon(\omega, -\mathbf{r}) &= \varepsilon_0 + \frac{1}{\pi} \, \mathrm{v.p.} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, \varepsilon(\omega', -\mathbf{r})}{\omega' - \omega} \, \mathrm{d}\omega' \\ &= \varepsilon_0 - \frac{1}{\pi} \, \mathrm{v.p.} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, \varepsilon(\omega', \mathbf{r})}{\omega' - \omega} \, \mathrm{d}\omega' \,. \end{aligned} \tag{39}$$

Expressions (38a) and (39) differ only in signs at the integral. Therefore, the system PT-symmetry condition for the real part of the permittivity at any real frequencies can hold only in the trivial case of the vacuum, where

$$\frac{1}{\pi} \, \mathrm{v.p.} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, \varepsilon(\omega', \mathbf{r})}{\omega' - \omega} \, \mathrm{d}\omega' \equiv 0 \,, \tag{40}$$

i.e., in the absence of losses in the entire frequency range: $\mathrm{Im}\, \varepsilon(\omega, \mathbf{r}) \equiv 0$. Therefore, the PT-symmetry condition cannot hold in the entire frequency range.

The formal proof presented above becomes obvious if we recall that condition (38) means that the permittivity $\varepsilon_{\rm gain}(\omega)$ of the gain medium for real frequencies must be equal to the complex conjugate permittivity $\varepsilon_{\rm loss}(\omega)$ of the passive component:

$$\varepsilon_{\rm gain}(\omega) = \varepsilon^*_{\rm loss}(\omega)\,.$$

The question arises about the behavior of $\varepsilon_{\rm gain}(\omega)$ defined in such a way in the upper half-plane of complex frequencies. We cannot simply set $\varepsilon_{\rm gain}(\omega) = \varepsilon^*_{\rm loss}(\omega)$, because $\varepsilon^*_{\rm loss}(\omega)$ is not an analytic function. Indeed, in passing from $\varepsilon_{\rm loss}(\omega)$ to $\varepsilon^*_{\rm loss}(\omega)$, the imaginary part $\varepsilon^*_{\rm loss}(\omega)$ changes its sign and the Cauchy–Riemann conditions are no longer satisfied. The choice $\varepsilon_{\rm gain}(\omega) = \varepsilon^*_{\rm loss}(\omega^*)$ is also inappropriate: although $\varepsilon^*_{\rm loss}(\omega^*)$ is an analytic function, it has singularities in the upper half-plane and is not a causal function. According to the continuity theorem [44], if two analytic functions coincide on a set having a limit point, they are equal. Therefore, an analytic continuation, different from $\varepsilon^*_{\rm loss}(\omega^*)$, from the real axis to the upper half-plane does not exist, and the system PT-symmetry condition can be satisfied neither on the entire real frequency axis nor in any finite frequency range. Indeed, according to the continuity theorem [44], the analytic continuation of $\varepsilon(\omega, \mathbf{r})$ from a finite frequency interval on the real axis to the entire complex frequency plane coincides with the analytic continuation of $\varepsilon(\omega, \mathbf{r})$ from the entire real axis to the complex plane, i.e., with $\varepsilon^*_{\rm loss}(\omega^*)$. The latter, as was said, does not satisfy the causality principle. Therefore, the system PT-symmetry condition can be satisfied only for a discrete frequency set.

We illustrate the last statement by the example of a medium with a permittivity with the Lorentzian dispersion

$$\varepsilon_{\rm loss}(\omega) = \varepsilon_{\rm mat} - \frac{\alpha}{\omega^2 - \omega_0^2 + 2{\rm i}\gamma\omega}\,, \tag{41}$$

where $\varepsilon_{\rm mat}$ is the permittivity of a matrix doped with amplifying (absorbing) components, γ is the gain (absorption) linewidth, α is the gain (absorption) coefficient, and ω_0 is the line frequency.

In the case of an absorbing medium, ${\rm Im}\,\varepsilon(\omega) > 0$, the parameters α and γ should be positive. Amplification corresponds to a negative imaginary part of the permittivity (${\rm Im}\,\varepsilon(\omega) < 0$), and, to satisfy the Kramers–Kronig relations, it is necessary to take $\alpha < 0$ and $\gamma > 0$, which ensures the causality of permittivity (41) [45, 46].[3]

However, the choice of permittivity in (41) with $\alpha < 0$, $\gamma > 0$ is incompatible with the requirement of the PT-symmetry of the system in a finite frequency interval [43]. In this case,

$$\begin{aligned}
{\rm Re}\,\varepsilon_{\rm loss}(\omega) &= \varepsilon_{\rm mat} - \frac{|\alpha|(\omega^2-\omega_0^2)}{(\omega^2-\omega_0^2)+4\gamma^2\omega^2}\,,\\
{\rm Im}\,\varepsilon_{\rm loss}(\omega) &= -\frac{2{\rm i}|\alpha|}{(\omega^2-\omega_0^2)+4\gamma^2\omega^2}\,,
\end{aligned} \tag{42a}$$

$$\begin{aligned}
{\rm Re}\,\varepsilon_{\rm gain}(\omega) &= \varepsilon_{\rm mat} + \frac{|\alpha|(\omega^2-\omega_0^2)}{(\omega^2-\omega_0^2)+4\gamma^2\omega^2}\,,\\
{\rm Im}\,\varepsilon_{\rm loss}(\omega) &= \frac{2{\rm i}|\alpha|}{(\omega^2-\omega_0^2)+4\gamma^2\omega^2}\,.
\end{aligned} \tag{42b}$$

[3] The choice $\alpha > 0$, $\gamma < 0$ means that $\varepsilon_{\rm gain}(\omega) = \varepsilon^*_{\rm loss}(\omega)$.

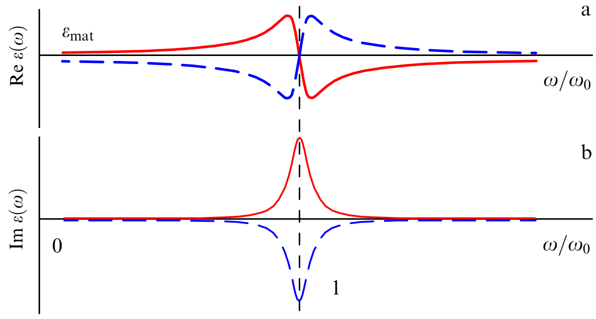


Figure 1. (a) Real and (b) imaginary parts of the permittivity for absorbing (solid curves) and gain (dashed curves) media in the case where imaginary parts differ only in sign: ${\rm Im}\,\varepsilon_{\rm loss} = -{\rm Im}\,\varepsilon_{\rm gain}$.

In other words, condition (37b) is fulfilled, whereas (37a) is not (Fig. 1).

We summarize the results in this section. The PT-symmetry condition for an optical system can be fulfilled only for a discrete frequency set and cannot be fulfilled in any finite frequency range.[4] Therefore, there is no sense to talk about a phase transition with frequency changing.

Below, we consider optical phenomena attributed in the literature to PT-symmetric systems and study how deviations from conditions (37) affect the predictions made for structures with exact PT symmetry.

4. Two-dimensional PT-symmetric optical systems

4.1 Analogy between the two-dimensional Helmholtz equation and the one-dimensional Schrödinger equation

Below, we study optical systems with the permittivity depending only on the coordinate x. We consider an electromagnetic wave with the amplitude $E(x)$, linearly polarized along the y axis and propagating in such a system. In this case, the problem of wave propagation reduces to a scalar problem [40], and, to find the field distribution, we have to solve the Helmholtz equation

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 \varepsilon(x) E = 0\,. \tag{43}$$

Because the permittivity is independent of the coordinate z, we can seek a solution of Eqn (43) as the product of two functions [40],

$$E(x,z) = g(z) f(x)\,, \tag{44}$$

which can be found from the equations [40]

$$\frac{\partial^2 f(x)}{\partial x^2} + \left(\frac{\omega}{c}\right)^2 \varepsilon(x) f(x) = k_z^2 f(x)\,, \tag{45a}$$

$$\frac{\partial^2 g(z)}{\partial z^2} + k_z^2 g(z) = 0\,. \tag{45b}$$

[4] The condition of the Hamiltonian pseudo-Hermiticity, which in optics is a generalization of the PT-symmetry condition (see Appendix 2), can also be fulfilled only for a discrete frequency set and cannot be fulfilled in any finite frequency range (see Appendix 3).

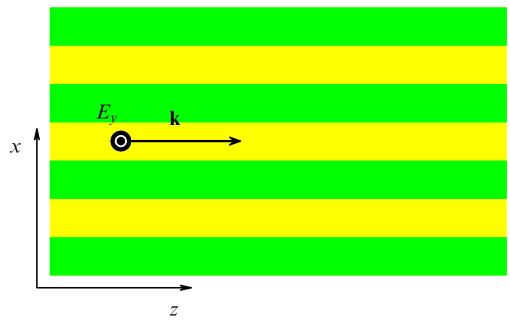


Figure 2. Two-dimensional PT-symmetric optical systems, $\varepsilon(x, z) = \varepsilon^*(-x, z)$. Different values of the permittivity are indicated by different colors; the arrow shows the propagation direction of electromagnetic waves.

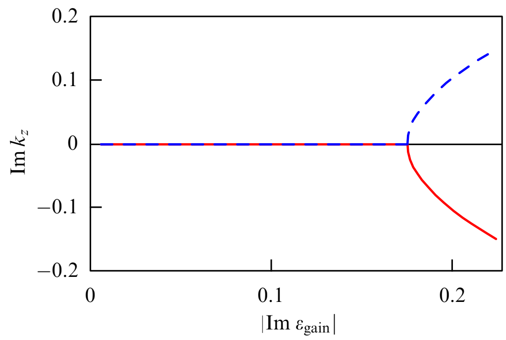


Figure 3. Dependence of the imaginary part of the effective wave vector on the imaginary part of the permittivity for $k_0 d = 10\pi$.

From Eqn (45b), we obtain

$$g(z) = A_1 \exp(\mathrm{i}k_z z) + A_2 \exp(-\mathrm{i}k_z z), \tag{46}$$

where the coefficients A_1 and A_2 are determined from the boundary conditions, and the separation constant k_z^2 is an eigenvalue of Eqn (45a).

Equation (45a) coincides with the one-dimensional stationary Schrödinger equation with Hamiltonian (21) in which $p^2/(2m) \to \partial^2/\partial x^2$, $V(x) \to (\omega/c)^2 \varepsilon(x)$, $E \to k_z^2$, and $\psi(x) \to f(x)$. This allows extending the results obtained in the framework of PT-symmetric quantum mechanics to optics. In this case, the operator $\hat{P}$ reduces to inversion along the x axis, and the operator $\hat{T}$ to complex conjugation.

4.2 Phase transition in two-dimensional PT-symmetric optical systems

We consider a PT-symmetric one-dimensional photonic crystal with a unit cell consisting of two layers with the same thickness d. The x axis is directed perpendicular to the layers (Fig. 2), the real part of the permittivity is the same in all layers, while the imaginary parts of the permittivity in neighboring layers differ in sign:

$$\varepsilon_n = \varepsilon_{\mathrm{R}} + (-1)^n \mathrm{i}\varepsilon_{\mathrm{I}}, \quad \varepsilon_{\mathrm{R}}, \varepsilon_{\mathrm{I}} > 0, \quad \varepsilon_{\mathrm{R}}, \varepsilon_{\mathrm{I}} \in \mathbb{R}, \tag{47}$$

where $n = 1, 2$ is the layer number in the unit cell. We consider an electromagnetic wave $E_y(x, z) = f(x) \exp(\mathrm{i}k_{\mathrm{eff}} z)$ linearly polarized along the y axis and propagating along the z axis (see Fig. 2), where k_{eff} is the effective wave vector along the wave propagation direction. The dispersion equation for k_{eff}^2 in this system reduces to two equations [47, 48]

$$\tan\frac{\beta_1 d}{2} \cot\frac{\beta_{\mathrm{g}} d}{2} = -\frac{\beta_1}{\beta_{\mathrm{g}}}, \tag{48a}$$

$$\tan\frac{\beta_1 d}{2} \cot\frac{\beta_{\mathrm{g}} d}{2} = -\frac{\beta_{\mathrm{g}}}{\beta_1}, \tag{48b}$$

where

$$\beta_1 = \sqrt{k_0^2(\varepsilon_{\mathrm{R}} + \mathrm{i}\varepsilon_{\mathrm{I}}) - k_{\mathrm{eff}}^2},$$

$$\beta_{\mathrm{g}} = \sqrt{k_0^2(\varepsilon_{\mathrm{R}} - \mathrm{i}\varepsilon_{\mathrm{I}}) - k_{\mathrm{eff}}^2}, \qquad k_0^2 = \left(\frac{\omega}{c}\right)^2.$$

Equation (48a) corresponds to a rapidly decaying wave [47, 48] even in a medium without losses and gain, and we do not consider this case here. Equation (48b) corresponds to the field distribution $f(x)$ in the unit cell of the form [47, 48]

$$f(x) = \begin{cases} \cos(\beta_1 x) + \dfrac{\cos(\beta_{\mathrm{g}} d) - \cos(\beta_1 d)}{(\beta_1/\beta_{\mathrm{g}}) \sin(\beta_{\mathrm{g}} d) + \sin(\beta_1 d)} \sin(\beta_1 x), & x \in [0, d], \\ \cos(\beta_{\mathrm{g}} x) + \dfrac{\cos(\beta_{\mathrm{g}} d) - \cos(\beta_1 d)}{\sin(\beta_{\mathrm{g}} d) + (\beta_{\mathrm{g}}/\beta_1) \sin(\beta_1 d)} \sin(\beta_{\mathrm{g}} x), & x \in [-d, 0), \end{cases} \tag{49}$$

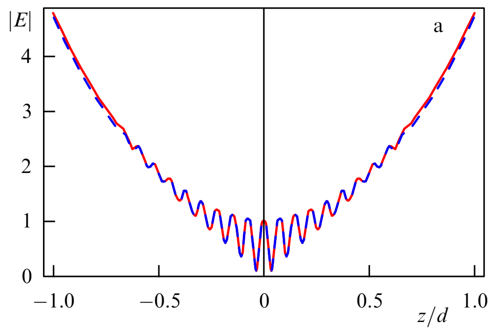
where $x = 0$ determines the plane separating layers inside the unit cell.

It follows from Eqn (48b) that for the imaginary parts ε_{I} of the permittivity smaller than a critical value $\varepsilon_{\mathrm{I}}^{\mathrm{cr}}(k_0 d)$, there are two eigensolutions with different real wave vectors k_{eff} for any frequency. For the imaginary parts of the permittivity exceeding the critical value, the wave vectors k_{eff} become complex (Fig. 3), their real parts coincide, while imaginary parts differ in sign. For $\varepsilon_{\mathrm{I}} < \varepsilon_{\mathrm{I}}^{\mathrm{cr}}$, the field distribution $f(x)$ is PT symmetric (Fig. 4a) and for $\varepsilon_{\mathrm{I}} > \varepsilon_{\mathrm{I}}^{\mathrm{cr}}$, the field distribution in the system is no longer PT symmetric (Fig. 4b).

Thus, passing from a real wave vector to a complex one is accompanied by a change in the symmetry of the eigensolutions $f(x)$: PT-symmetric solutions are changed to PT-nonsymmetric solutions (see Fig. 4). Therefore, a phase transition is observed in the system, which is caused by the spontaneous PT-symmetry breaking for the solution $f(x)$ upon increasing the modulus of the imaginary part ε_{I} of the permittivity. Below the phase transition point, the eigensolutions are PT symmetric. The field amplitude in a layer with losses is equal to the field amplitude in a gain layer (Fig. 4a). As a result, the energy dissipation in the layer with losses is compensated by the energy influx in the gain layer, $\mathrm{Im}\,\varepsilon_{\mathrm{gain}}|E_{\mathrm{gain}}|^2 + \mathrm{Im}\,\varepsilon_{\mathrm{loss}}|E_{\mathrm{loss}}|^2 = 0$. In other words, the field energy is conserved, which corresponds to a real value of the wave vector. Above the phase transition point, the eigensolutions are PT nonsymmetric, the field amplitudes in the loss layer and the gain layer are not equal (Fig. 4b), and $\mathrm{Im}\,\varepsilon_{\mathrm{gain}}|E_{\mathrm{gain}}|^2 + \mathrm{Im}\,\varepsilon_{\mathrm{loss}}|E_{\mathrm{loss}}|^2 \neq 0$. The field decay in the loss layer is no longer compensated by the field increase in the gain layer. As a result, the electromagnetic field can propagate in the system with decay or amplification.

4.3 'Loss-induced transparency'

To produce a PT-symmetric system, it is necessary to ensure the exact coincidence of the moduli of imaginary parts of the permittivity in amplifying and absorbing media. In real



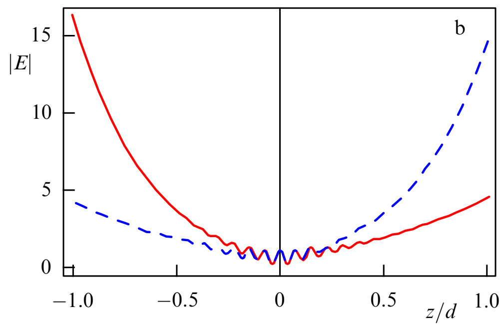


Figure 4. Field distribution in a system of two layers of the thickness $k_0 d = 10\pi$ with the permittivity $\varepsilon_n = \varepsilon_R + (-1)^n \mathrm{i}\varepsilon_I$, $\varepsilon_R > 0$, $\varepsilon_R, \varepsilon_I \in \mathbb{R}$, where n is the layer number, $n = 1, 2$: (a) below the phase transition point, $\varepsilon_I = 0.15 < \varepsilon_I^{cr} \approx 0.18$; (b) above the phase transition point, $\varepsilon_I = 0.35 > \varepsilon_I^{cr} \approx 0.18$. The solid and dashed curves correspond to amplitudes of two different eigenmodes of the system.

optical systems, this is difficult to achieve, and controlling these quantities with relation (37b) preserved is even more difficult.

However, it was found in [49] that the phase transition described in Section 4.2 could be observed in systems that are not PT symmetric. This is possible if the description of such systems can be formally reduced to the description of an auxiliary PT-symmetric system.

We consider an infinite optical system for which the real part $\mathrm{Re}\,\varepsilon(\omega, x)$ of the permittivity is an even function of the coordinate and the imaginary part $\mathrm{Im}\,\varepsilon(\omega, x)$ is the sum of an odd function of coordinate $\varepsilon_I(\omega, x)$ and a constant C:

$$\mathrm{Re}\,\varepsilon(\omega, x) = \mathrm{Re}\,\varepsilon(\omega, -x)\,, \tag{50a}$$

$$\mathrm{Im}\,\varepsilon(\omega, x) = \varepsilon_I(\omega, x, \mu) + C\,, \tag{50b}$$

$$\varepsilon_I(\omega, x) = -\varepsilon_I(\omega, -x)\,. \tag{50c}$$

We note that the dependence on z is absent. Below, we assume that $\varepsilon_I(\omega, x, \mu)$ linearly depends on a real factor μ parameterizing system (50), $\varepsilon_I(\omega, x, \mu) = \mu\varepsilon_I(\omega, x)$. Such a system is not PT-symmetric because of the breaking of condition (37b). The addition of a constant means that the system can be purely dissipative, which is much simpler to realize experimentally.

We describe the transformation that reduces system (50) to a PT-symmetric system. As in the derivation of (43)–(46), we seek the field distribution in the system in the form

$$E(x, z) = \exp(\mathrm{i}kz)\, f(x)\,. \tag{51}$$

We introduce the new variable

$$e(x, z) = E(x, z)\exp(\alpha z)\,. \tag{52}$$

Then the Helmholtz equation (43) for $e(x, z)$ can be represented in the form

$$\frac{\partial^2 e(x,z)}{\partial x^2} + \frac{\partial^2 e(x,z)}{\partial z^2} + 2\alpha\frac{\partial e(x,z)}{\partial z} + \left(\frac{\omega}{c}\right)^2\left(\mathrm{Re}\,\varepsilon(\omega, x) + \frac{\alpha^2}{(\omega/c)^2} + \mathrm{i}\varepsilon_I(\omega, x, \mu) + \mathrm{i}C\right)e(x,z) = 0\,. \tag{53}$$

Because the permittivity is independent of z, we can seek a solution of Eqn (53) in the form of the product of two functions [40] (see Section 4.1):

$$e(x, z) = \exp(\mathrm{i}k_{eR} z + k_{eI} z)\, f(x)\,. \tag{54}$$

The term with the first derivative is then eliminated from Eqn (53), because $\partial e(x,z)/\partial z = (\mathrm{i}k_{eR} + k_{eI})\, e(x,z)$, and Eqn (53) takes the form

$$\frac{\partial^2 e(x,z)}{\partial x^2} + \frac{\partial^2 e(x,z)}{\partial z^2} + \frac{\omega^2}{c^2}\left(\mathrm{Re}\,\varepsilon(\omega, x) + \frac{\alpha^2}{(\omega/c)^2} + \frac{2\alpha k_{eI}}{(\omega/c)^2} + \mathrm{i}\varepsilon_I(\omega, x, \mu) + \mathrm{i}C + \mathrm{i}\,\frac{2\alpha k_{eR}}{(\omega/c)^2}\right)e(x,z) = 0\,. \tag{55}$$

Equation (55) is a standard wave equation with the permittivity

$$\varepsilon_{\mathrm{eff}}(x) = \left(\mathrm{Re}\,\varepsilon(\omega, x) + \frac{\alpha^2}{(\omega/c)^2} + \frac{2\alpha k_{eI}}{(\omega/c)^2}\right) + \mathrm{i}\left(\varepsilon_I(\omega, x, \mu) + C + \frac{2\alpha k_{eR}}{(\omega/c)^2}\right).$$

We choose the parameter α in transformation (52) such that the coordinate-independent imaginary part of the permittivity vanishes: $C + (2\alpha c^2 k_{eR})/\omega^2 = 0$. As a result, we obtain a system with the effective permittivity

$$\varepsilon_{\mathrm{eff}}(x) = \left(\mathrm{Re}\,\varepsilon(x) + \frac{C^2}{4}\,\frac{(\omega/c)^2}{k_{eR}^2} - C\,\frac{k_{eI}}{k_{eR}}\right) + \mathrm{i}\varepsilon_I(x)\,, \tag{56}$$

satisfying the PT-symmetry condition (37). The terms in (56) depending on the wave vector $k_e = k_{eR} + \mathrm{i}k_{eI}$ are PT symmetric because the wave vectors do not change under the simultaneous action of the spatial inversion operator $\hat{P}$ and the time reversal operator $\hat{T}$. Therefore, the effective permittivity $\varepsilon_{\mathrm{eff}}(x)$ is PT symmetric and eigensolutions of Eqn (55) can change their symmetry upon changing the parameter μ.

Because the electric field $E(x, z)$ and the function $e(x, z)$ are connected with each other in a one-to-one manner [see (52)], the change in the symmetry of the auxiliary field $e(x, z)$ leads to a change in the real field distribution $E(x, z)$.[5] Thus,

[5] A change in the electric field distribution $E(x, z)$ upon varying the imaginary part of the permittivity corresponds to the phase transition from P-symmetric eigensolutions to P-nonsymmetric eigensolutions. We recall that the operator $\hat{P}$ is here assumed to reduce to inversion only along the x axis.

we pointed to the possibility of observing a 'hidden' phase transition with a change in the solution symmetry in systems for which the PT-symmetry condition is broken.

In [49], a phase transition in a system obeying conditions (50) was found experimentally and studied theoretically. The authors of [49] considered a system of two coupled optical waveguides filled with materials with permittivities ε_1 and ε_2. The first material had neither losses nor amplification ($\mathrm{Im}\,\varepsilon_1 = 0$), while dissipation occurred in the second material ($\mathrm{Im}\,\varepsilon_2 > 0$). This system was analyzed in [49] in the interacting mode approximation [25, 49–51]. The optical field in waveguides was sought in the form $E_n = U_n(z)F_n(x, y)$, where $F_n(x, y)$ describes the field distribution in the cross section of the nth waveguide and $U_n(z)$ is the 'amplitude'. In the interacting mode approximation, $U_n(z)$ obeys the equations [49]

$$\mathrm{i}\,\frac{\mathrm{d}U_1}{\mathrm{d}z} = \beta U_1 + \kappa U_2\,, \tag{57a}$$

$$\mathrm{i}\,\frac{\mathrm{d}U_2}{\mathrm{d}z} = (\beta + \delta\beta)U_2 + \kappa^* U_1\,, \tag{57b}$$

where β is the propagation constant in the system of waveguides without losses or amplification ($\mathrm{Im}\,\varepsilon_1 = \mathrm{Im}\,\varepsilon_2 = 0$), and κ is the coefficient of interaction between waveguides. A change $\delta\beta$ in the propagation constant in the system of waveguides produced by the addition of losses is proportional to the imaginary part of the permittivity. In such a system, two guided eigenmodes exist with propagation constants $\beta_\pm = \beta + \delta\beta/2 \pm [\kappa^*\kappa - (\delta\beta/2)^2]^{1/2}$. For values of $\mathrm{Im}\,\varepsilon_2$ lower than a threshold value, the eigenmodes of the waveguides are spatially symmetric, and the propagation constant decreases with increasing losses ($\mathrm{Im}\,\varepsilon_2$) (Fig. 5). When $\mathrm{Im}\,\varepsilon_2$ increases to the threshold, the waveguide eigenmodes become spatially nonsymmetric (Fig. 4b) and, as $\mathrm{Im}\,\varepsilon_2$ increases further, the propagation coefficient for one of the modes in the system increases (see the curve in Fig. 5). The increase in the propagation coefficient is explained by the fact that as $\mathrm{Im}\,\varepsilon_2$ increases, the field in the eigenmode of the system is mainly concentrated in the waveguide without losses.

The effect of the increase in the propagation coefficient with increasing losses was called 'the loss-induced transparency' [49].

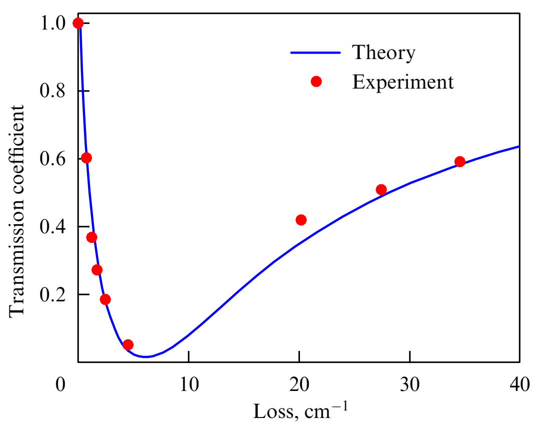


Figure 5. Transmission coefficient for light propagating through a system of two coupled waveguides as a function of the loss in the second waveguide [49].

4.4 Refraction asymmetry in a PT-symmetric optical system

The authors of [26] assigned the property of asymmetric refraction to PT-symmetric systems: when a linearly polarized plane wave is incident from a vacuum at an arbitrary angle θ on a semi-infinite PT-symmetric system, the field distribution inside a photonic crystal is not mapped into its P-symmetric image upon changing the angle of incidence $\theta \to -\theta$:

$$E(x, z, \theta) \neq \hat{P}E(x, z, \theta) = E(-x, z, -\theta)\,. \tag{58}$$

The authors of [26] considered a semi-infinite photonic crystal bordering a vacuum over the $z = 0$ plane, in which the permittivity varies along the x axis. Is was assumed that the real and imaginary parts of the permittivity distribution change asynchronously:[6]

$$\varepsilon(x) = \varepsilon_0 + \varepsilon_1 \cos(kx) + \mathrm{i}\varepsilon_2 \sin(kx)\,. \tag{59}$$

Obviously, such a system is PT-symmetric. When a plane wave linearly polarized along the y axis is incident on the vacuum–photonic-crystal interface, the field in the system can be sought in the form [40]

$$E(x,z) = \exp(\mathrm{i}k_z^{\mathrm{PC}} z)\exp(\mathrm{i}k_x^{\mathrm{inc}} x)\sum_{n=-\infty}^{+\infty} C_n \exp(\mathrm{i}k_\varepsilon n x)\,, \quad z < 0\,, \tag{60a}$$

$$E(x,z) = \exp\left[\mathrm{i}\left(k_x^{\mathrm{inc}} x + \sqrt{k_0^2 - (k_x^{\mathrm{inc}})^2}\, z\right)\right] + \exp(\mathrm{i}k_x^{\mathrm{inc}} x) \times \sum_{n=-\infty}^{+\infty} R_n \exp\left[\mathrm{i}k_\varepsilon n x - \sqrt{k_0^2 - (k_\varepsilon n + k_x^{\mathrm{inc}})^2}\, z\right], \quad z \geqslant 0\,, \tag{60b}$$

where k_z^{PC} is the wave vector in the photonic crystal in the z direction perpendicular to the photonic crystal surface, k_x^{inc} is the tangential component of the wave vector of the incident wave, and $k_\varepsilon = 2\pi/d$ is the reciprocal lattice vector. Substituting field distribution (60) and permittivity (59) in the Helmholtz equation

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 \varepsilon(x) E = 0\,, \tag{61}$$

we obtain an infinite system of equations for Fourier expansion coefficients $C_n(k_x^{\mathrm{inc}})$:

$$\left[k_0^2 - (k_z^{\mathrm{PC}})^2 - (k_x^{\mathrm{inc}} + k_\varepsilon n)^2\right] C_n(k_x^{\mathrm{inc}}) + \frac{k_1^2 + k_2^2}{2}\, C_{n-1}(k_x^{\mathrm{inc}}) + \frac{k_1^2 - k_2^2}{2}\, C_{n+1}(k_x^{\mathrm{inc}}) = 0\,, \tag{62}$$

where $k_0^2 = (\omega/c)^2\varepsilon_0$, $k_1^2 = (\omega/c)^2\varepsilon_1$, $k_2^2 = (\omega/c)^2\varepsilon_2$, and k_z is found as an eigenvalue of boundary value problem (62) [40, 41]. The refraction asymmetry condition (58)

$$E(x, z, k_x^{\mathrm{inc}}) = \exp(\mathrm{i}k_z^{\mathrm{PC}} z)\exp(\mathrm{i}k_x^{\mathrm{inc}} x)\sum_{n=-\infty}^{+\infty} C_n \exp(\mathrm{i}k_\varepsilon n x) \neq E(-x, z, -k_x^{\mathrm{inc}}) = \exp(\mathrm{i}k_z^{\mathrm{PC}} z)\exp(\mathrm{i}k_x^{\mathrm{inc}} x) \times \sum_{n=-\infty}^{+\infty} C_{-n} \exp(\mathrm{i}k_\varepsilon n x) \tag{63}$$

[6] We consider the case of a nonmagnetic medium in which the magnetic permeability is unity.

can be represented in terms of Fourier expansion coefficients as

$$C_n(k_x^{\mathrm{inc}}) \neq C_{-n}(-k_x^{\mathrm{inc}})\,. \tag{64}$$

A system of equations for $C_{-n}(-k_x^{\mathrm{inc}})$ is obtained from system of equations (62) by the substitutions $k_x \to -k_x$ and $n \to -n$:

$$\left[k_0^2 - k_z^2 - (k_x^{\mathrm{inc}} + k_\varepsilon n)^2\right] C_{-n}(-k_x^{\mathrm{inc}}) + \frac{k_1^2 + k_2^2}{2}\, C_{-n-1}(-k_x^{\mathrm{inc}}) + \frac{k_1^2 - k_2^2}{2}\, C_{-n+1}(-k_x^{\mathrm{inc}}) = 0\,. \tag{65}$$

We can see from Eqns (62) and (65) that the coefficients $C_n(k_x^{\mathrm{inc}})$ and $C_{-n}(-k_x^{\mathrm{inc}})$ are the same, whereas $C_{n-1}(k_x^{\mathrm{inc}})$ and $C_{-(n-1)}(-k_x^{\mathrm{inc}})$ differ. Therefore, in the PT-symmetric system under study, where $k_1^2 \neq 0$ and $k_2^2 \neq 0$, refraction is asymmetric.[7]

It is important to note that the refraction asymmetry is not related to the PT-symmetry of the system and can be observed in any P-nonsymmetric system. Indeed, by definition, refraction is called symmetric if the field distribution in the system is transformed into its P-symmetric image:

$$\hat{P}E(x, z, k_x) = E(-x, z, -k_x) = E(x, z, k_x)\,. \tag{66}$$

If Helmholtz equation (61) is invariant under the $\hat{P}$ transformation and $E(x, z, k_x)$ is an eigensolution of the Helmholtz equation, then $PE(x, z, k_x)$ is also an eigensolution of the Helmholtz equation, and hence refraction in the system is symmetric. If the optical system is not P-symmetric, then $\hat{P}E(x, z, k_x)$ is not an eigensolution of the Helmholtz equation and refraction is asymmetric.

The system considered in [26] is not P-symmetric, $\varepsilon(x) \neq \varepsilon(-x)$; therefore, there is no surprise that asymmetric refraction is observed in it.

5. One-dimensional PT-symmetric optical systems

In Section 4, we considered two-dimensional PT-symmetric optical systems for which Eqn (45a) is equivalent to the one-dimensional stationary Schrödinger equation. We saw that this allows extending the results obtained in PT-symmetric quantum mechanics to optics. In particular, in two-dimensional PT-symmetric optical systems (37), a phase transition can occur due to PT-symmetry breaking for eigensolutions of the Helmholtz equation.

Although it is impossible to establish a direct analogy with the Schrödinger equation in the one-dimensional case, the PT-symmetry of an optical system still implies the condition $\varepsilon(x) = \varepsilon^*(-x)$. In real optical systems, this condition can be satisfied only for a discrete frequency set (see Section 3.2). Therefore, it is reasonable to consider the properties of the scattering matrix of a PT-symmetric optical system only at one real frequency ω selected in advance.

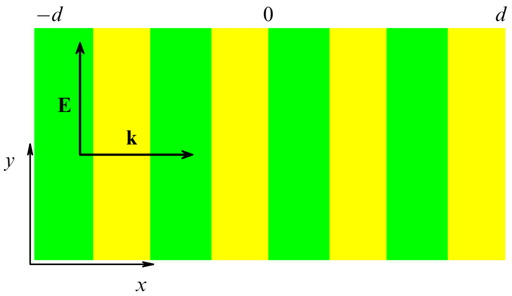


Figure. 6. (Color online.) One-dimensional PT-symmetric optical systems, $\varepsilon(x) = \varepsilon^*(-x)$. Different values of the imaginary part of the permittivity are shown by green and yellow colors. The vector **E** shows polarization and the vector **k**, the propagation direction of an electromagnetic wave.

5.1 Phase transition in a one-dimensional optical system

For definiteness, we assume that light propagates along the direction x of variation in the permittivity and is polarized along the y axis [53–55], while a PT-symmetric system occupying the region above the coordinate x from $-d$ to d is surrounded by the vacuum from both sides (Fig. 6). Only one component of the electric field $E_y = E(x)$ satisfies the Helmholtz equation [53, 54]:

$$\frac{\partial^2 E(x)}{\partial x^2} + \left(\frac{\omega}{c}\right)^2 \varepsilon(x, \omega)\, E(x) = 0\,. \tag{67}$$

Here, unlike in two-dimensional case (47), the wave propagation direction and the permittivity changing direction coincide. Such a system can be conveniently described using the scattering matrix s [53]. The scattering matrix s relates the amplitudes of incident (ϕ_k) and scattered (ψ_n) waves:

$$\psi_n = s_{nk}\phi_k\,, \tag{68}$$

where $n, k = 1, 2$, the subscript 1 corresponds to a plane wave incident on the system from the left (ψ_1 describes the reflected and transmitted waves), and the subscript 2 refers to a plane wave incident from the right.

In two-dimensional PT-symmetric optical systems, we explained the phase transition by the change in the symmetry of eigenmodes in the system, but in the one-dimensional case, we consider the change in the symmetry of eigenvectors of the scattering matrix s [53].

The electric field outside a PT-symmetric system is equal to the sum of the incident and scattered waves:

$$E(x, \omega) = \begin{cases} \phi_1 \exp\left(\mathrm{i}\,\dfrac{\omega}{c}\,x\right) + \psi_1 \exp\left(-\mathrm{i}\,\dfrac{\omega}{c}\,x\right), & x < -d\,, \\ \phi_2 \exp\left(-\mathrm{i}\,\dfrac{\omega}{c}\,x\right) + \psi_2 \exp\left(\mathrm{i}\,\dfrac{\omega}{c}\,x\right), & x > d\,. \end{cases} \tag{69}$$

To determine the properties of the scattering matrix in PT-symmetric optical systems, we apply the operator $\hat{P}\hat{T}$ to the field distribution in system (69). The action of $\hat{P}$ here reduces to the substitution $x \to -x$, and the action of $\hat{T}$ leads to complex conjugation, $\mathrm{i} \to -\mathrm{i}$. Hence, $\mathrm{i}x \to \mathrm{i}x$ under the

[7] In the literature, instead of the term 'asymmetric refraction', the term 'nonreciprocal propagation' is commonly used, which seems inappropriate to us because this term usually means the difference in the propagation constants in the forward and backward directions. This is possible only in nonlinear or gyrotropic media or in media with the propagation coefficient varying in time [42, 52]. The propagation nonreciprocity term used in the study of PT-symmetric systems means a difference in refraction of light incident on a PT-symmetric system at angles $\pm\theta$, which is not eliminated by the mirror mapping of the system with respect to the perpendicular to the surface [3], i.e., the refraction asymmetry, $E(x, z, k_x) \neq E(-x, z, -k_x)$.

action of $\hat{P}\hat{T}$, and Eqn (69) changes as (see Section 2.1)

$$\hat{P}\hat{T}E(x,\omega) = \begin{cases} (\hat{P}\hat{T}\phi_1)\exp\left(\mathrm{i}\,\dfrac{\omega}{c}\,x\right) + (\hat{P}\hat{T}\psi_1)\exp\left(-\mathrm{i}\,\dfrac{\omega}{c}\,x\right), & x > d\,, \\ (\hat{P}\hat{T}\phi_2)\exp\left(-\mathrm{i}\,\dfrac{\omega}{c}\,x\right) + (\hat{P}\hat{T}\psi_2)\exp\left(\mathrm{i}\,\dfrac{\omega}{c}\,x\right), & x < -d\,. \end{cases} \tag{70}$$

Because Helmholtz equation (67) for a PT-symmetric system $(\varepsilon(x) = \varepsilon^*(-x))$ is invariant under the PT transformation and (69) is a solution of the Helmholtz equation, (70) is a solution of the Helmholtz equation. By comparing (69) and (70), we see that $\hat{P}\hat{T}\psi_n$ now plays the role of amplitudes of plane waves incident on the system, while $\hat{P}\hat{T}\phi_n$ are the amplitudes of plane waves scattered from the system. These amplitudes are connected with each other by the same scattering matrix as before the PT-transformation: $\hat{P}\hat{T}\phi_n = s_{nk}\hat{P}\hat{T}\psi_k$.[8] In other words,

$$\hat{P}\hat{T}\psi_n = s_{nk}^{-1}\hat{P}\hat{T}\phi_k\,. \tag{71}$$

Applying the operator $\hat{P}\hat{T}$ to Eqn (68) from the left, we obtain

$$\hat{P}\hat{T}\psi_n = (\hat{P}\hat{T}s_{nk}\hat{P}\hat{T})\,\hat{P}\hat{T}\phi_k\,. \tag{72}$$

It follows from (71) and (72) that the scattering matrix s in PT-symmetric systems is transformed as [53]

$$\hat{P}\hat{T}s_{nk}\hat{P}\hat{T} = s_{nk}^{-1}\,. \tag{73}$$

We introduce the eigenvectors ϕ_n^1 and ϕ_n^2 of the scattering matrix and the corresponding eigenvalues s^1 and s^2 $(s_{nk}\phi_k^i = s^{(i)}\phi_k^{(i)})$. Multiplying the left- and right-hand sides of Eqn (73) by a scattering matrix eigenvector ϕ_k^i $(i = \{1,2\})$, we obtain

$$\hat{P}\hat{T}s_{nk}(\hat{P}\hat{T}\phi_k^i) = s_{nk}^{-1}\phi_k^i = \frac{1}{s^{(i)}}\,\phi_k^{(i)}\,. \tag{74}$$

Now, applying the operator $\hat{P}\hat{T}$ to (74) from the left and taking into account that $(\hat{P}\hat{T})^2 = \hat{1}$, we find

$$s_{nk}(\hat{P}\hat{T}\phi_k^i) = (\hat{P}\hat{T}s_{nk}\phi_k^{(i)})\,\frac{1}{(s^{(i)})^*}\,(\hat{P}\hat{T}\phi_k^{(i)})\,. \tag{75}$$

Expanding $\hat{P}\hat{T}\phi_k^i$ with respect to the eigenvectors of the scattering matrix, we obtain a matrix representation of the operator $\hat{P}\hat{T}$ in the basis of ϕ_k^i:

$$\hat{P}\hat{T}\phi_k^i = c^{ij}\phi_k^j\,. \tag{76}$$

Because $(\hat{P}\hat{T})^2 = \hat{1}$ and $(\hat{P}\hat{T})^+ = \hat{P}\hat{T}$, it follows that $c^{ij}c^{jl} = \delta^{il}$, where δ^{il} is the Kronecker delta, and $(c^{ij})^* = c^{ji}$. As a result, the c^{ij} matrix can be represented either in the form (see Appendix 5)

$$c = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{77a}$$

or as

$$c = \begin{pmatrix} \cos\theta & \sin\theta\exp(\mathrm{i}\phi) \\ \sin\theta\exp(-\mathrm{i}\phi) & -\cos\theta \end{pmatrix}, \tag{77b}$$

where θ and ϕ are real numbers. For $\theta = \pi/2 + \pi n$, $n \in \mathbb{N}$, the second case reduces to

$$c = \begin{pmatrix} 0 & \exp(\mathrm{i}\phi) \\ \exp(-\mathrm{i}\phi) & 0 \end{pmatrix}. \tag{77c}$$

Substituting (76) in (75), we arrive at the system of equations for c_{ij}:

$$s_{nk}\sum_j c^{ij}\phi_k^j = \sum_j c^{ij}s^j\phi_n^j = \sum_j \frac{c^{ij}}{(s^{(i)})^*}\,\phi_n^j\,. \tag{78}$$

Below, we consider only reciprocal systems. Eigensolutions of the scattering matrix of a reciprocal system with different eigenvalues are orthogonal, $\sum_k (\phi_k^j)^{\mathrm{T}}\phi_k^l = \delta^{jl}$ (see Appendix 5). Taking the scalar product of Eqn (78) with $(\phi_k^l)^{\mathrm{T}}$, we arrive at the eigenvalue problem for the scattering matrix:

$$c^{il}s^{(l)} = \frac{c^{il}}{(s^{(i)})^*}\,. \tag{79}$$

Depending on the form of the c matrix, system of equations (79) has three possible solutions:

(1) if $c^{11} = \pm 1$, $c^{22} = \pm 1$, and $c^{12} = c^{21} = 0$ [case (77a)], the eigenvectors of the scattering matrix are PT-symmetric, $\hat{P}\hat{T}\phi_k^i = \pm\phi_k^i$, and the eigenvalues are $|s^i| = 1$;

(2) if $c^{11} = c^{22} = 0$, $c^{12} \neq 0$, and $c^{21} \neq 0$ [case (77c)], the eigenvectors of the scattering matrix are PT-nonsymmetric and are mapped into one another by the $\hat{P}\hat{T}$ transformation: $\hat{P}\hat{T}\phi_k^i = \phi_k^j$, $i \neq j$, while the scattering matrix eigenvalues are related as $s^i = 1/(s^j)^*$, $i \neq j$;

(3) if $c^{11} = -c^{22}$, $c^{12} \neq 0$, and $c^{21} \neq 0$ [case (77b)], the scattering matrix eigenvalues for different eigenvectors are equal, $s^i = s^j = 1/(s^i)^*$, $i \neq j$.

Below the phase transition point (the phase of PT-symmetric solutions), the eigenvectors of the scattering matrix are PT-symmetric, $\hat{P}\hat{T}\phi_k^i = \pm\phi_k^i$, and the amplitudes of both eigenvalues matrix are unity, $|s^i| = 1$, corresponding to the first case. Above the phase transition point, the eigenvectors of the scattering matrix are PT-nonsymmetric, corresponding to the second case, $\hat{P}\hat{T}\phi_k^i = \pm\phi_k^j$, $i \neq j$, and the eigenvalues are related as $s^i = 1/(s^j)^*$, $i \neq j$ [53].

In the third case, degeneracy exists: the eigenvalues are equal to each other and the eigenvectors coincide (are linearly dependent). This point corresponds to the boundary between the region with PT-symmetric and PT-nonsymmetric eigensolutions, i.e., to the phase transition point.

To determine the condition under which the phase transition occurs from PT-symmetric eigenvectors of the scattering matrix to PT-nonsymmetric ones, we recall that the operator $\hat{P}\hat{T}$ and the Helmholtz operator in Eqn (67) commute with each other in any PT-symmetric system. In this case, if $\hat{P}\hat{T}$ were a linear operator, the eigensolutions of the Helmholtz equation would always remain PT-symmetric, i.e., would coincide with eigensolutions of the operator $\hat{P}\hat{T}$. In reality, the operator $\hat{P}\hat{T}$ is antilinear because its action includes complex conjugation (see Section 2.1), and therefore the eigensolutions of $\hat{P}\hat{T}$ and of

[8] A PT-symmetric system does not change after the PT-transformation: $PT\varepsilon(x) = \varepsilon^*(-x) = \varepsilon(x)$ and, because the scattering matrix is determined only by the structure of the system, the scattering matrix of a PT-symmetric system does not change after the PT transformation.

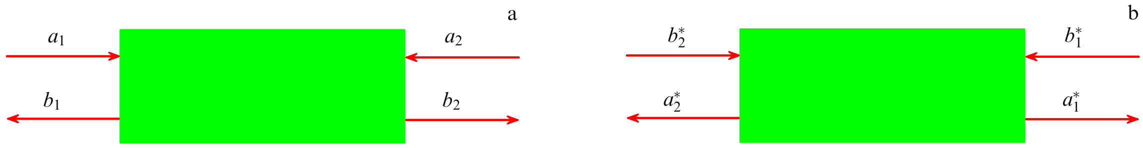


Figure 7. (a) Schematic representation of the scattering of plane waves from a one-dimensional optical system: a_i are the complex amplitudes of waves incident on the system, b_i are the complex amplitudes of waves scattered by the system. (b) Schematic representation of the scattering of plane waves from a one-dimensional optical system, shown in Fig. 7a, after its $\hat{P}\hat{T}$ transformation.

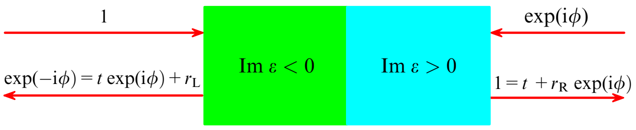


Figure 8. PT-symmetric field distribution in a one-dimensional optical system.

the Helmholtz operator may not coincide. In this case, the eigensolutions of Helmholtz equation (67) are PT-nonsymmetric (see Section 2.3).

As an example, we consider a one-dimensional system on which plane waves with amplitudes a_1 and a_2 are respectively incident from the left and right, while waves with amplitudes b_1 and b_2 are scattered from the system (Fig. 7a). By performing the $\hat{P}\hat{T}$ transformation of the field distribution, we see that time reversal leads to a change in the propagation direction of the waves and complex conjugation of amplitudes (16), while spatial inversion again causes a change in the propagation direction of the waves and the interchange of the waves propagating from left to right and from right to left [see expressions (69) and (70)]. As a result, a wave incident on the system from the left (right) is transformed into a wave scattered to the right (left) with the complex conjugate amplitude (Fig. 7b).

Hence, for the field distribution in the system to be PT-symmetric, the solution must not change after the PT transformation. This requires that the conditions $a_1 = b_2^*$ and $a_2 = b_1^*$ be satisfied. At the same time, because the scattering matrix connects the amplitudes of incident and scattered waves, the amplitudes of incident and scattered waves for eigenvectors of the matrix should have pairwise equal moduli, $|a_1| = |b_1|$ and $|a_2| = |b_2|$. As a result, the amplitudes of the waves incident from the left and right for PT-symmetric eigensolutions must be equal, $|a_1| = |a_2|$.

We consider the case where $a_1 = 1$ and $a_2 = \exp(\mathrm{i}\phi)$ in more detail, where ϕ is an arbitrary phase factor (Fig. 8). For the field distribution in the system to be PT-symmetric, the equalities $b_2 = 1$ and $b_1 = \exp(-\mathrm{i}\phi)$ must be satisfied.

At the same time, the plane wave scattered from the system to the left is the sum of the transmitted wave $t\exp(\mathrm{i}\phi)$ incident from the right (t is the transmission coefficient of the system)[9] and the reflected wave r_L incident from the left (r_L is the reflection coefficient of the system for light incident from the left). Finally, we obtain

$$\exp(-\mathrm{i}\phi) = t\exp(\mathrm{i}\phi) + r_\mathrm{L}\,. \tag{80a}$$

[9] We assume that the system is reciprocal, i.e., transmission coefficients for light propagating through the system from left to right and from right to left are equal.

The plane wave scattered from the system from the right is the sum of the transmitted wave t incident from the left and the reflected wave $r_\mathrm{R}\exp(\mathrm{i}\phi)$ incident from the right (r_R is the reflection coefficient of the system for light incident from the right), which gives

$$1 = t + r_\mathrm{R}\exp(\mathrm{i}\phi) \tag{80b}$$

for a PT-symmetric field distribution. Solving system of equations (80a), (80b), we obtain

$$\frac{r_\mathrm{L} - r_\mathrm{R}}{t} = -2\mathrm{i}\sin\phi\,, \tag{81}$$

and the eigenvectors of the scattering matrix have the form

$$\phi_k^1 = \begin{pmatrix} 1 \\ \exp\left[\mathrm{i}\arcsin\left(-\dfrac{r_\mathrm{L}-r_\mathrm{R}}{2\mathrm{i}t}\right)\right] \end{pmatrix}, \qquad \phi_k^2 = \begin{pmatrix} 1 \\ \exp\left\{\mathrm{i}\left[\pi - \arcsin\left(-\dfrac{r_\mathrm{L}-r_\mathrm{R}}{2\mathrm{i}t}\right)\right]\right\} \end{pmatrix}. \tag{82}$$

The modulus of the right-hand side of Eqn (81) cannot exceed two, and therefore the operator $\hat{P}\hat{T}$ has eigensolutions only if

$$\left|\frac{r_\mathrm{L} - r_\mathrm{R}}{t}\right| \leqslant 2\,. \tag{83}$$

Thus, the problem solution for the PT-symmetric system is PT-symmetric only if condition (83) is satisfied [53]. If condition (83) is violated, the phase transition from the PT-symmetric to PT-nonsymmetric eigenvectors of the scattering matrix occurs. At the point of the phase transition from the PT-symmetric to PT-nonsymmetric eigensolutions, $\arcsin[-(r_\mathrm{L} - r_\mathrm{R})/(2\mathrm{i}t)] = \pi/2$, and the eigenvectors of the scattering matrix coincide, $\phi_k^1 = \phi_k^2$.

5.2 Lasing in PT-symmetric optical systems and the 'hidden' phase transition in a one-dimensional optical system

So far, we have considered PT-symmetric systems without looking at the possibility of lasing in them. At the same time, any PT-symmetric system contains a gain medium. If the region occupied by the gain medium forms a resonator, lasing can occur in the system.

In the lasing regime, waves escaping from the system exist even in the absence of incident waves. To determine the amplitudes of the escaping waves and the lasing frequency, it is necessary to take the nonlinear dependence of the response function of the gain medium on the field amplitude into account. Nevertheless, the conditions for lasing to appear in the system can be determined in the linear description of the medium [46]. Within the linear approximation, lasing develops upon the transition of one of the poles of the scattering matrix to the upper half-plane of the complex frequency plane [46, 56, 57]. A pole of the scattering matrix corresponds to one of the eigenvalues of the scattering matrix turning into infinity. The amplitude of the escaping waves at the lasing onset point in the absence of incident waves is zero.

In the region where condition (83) is violated, one of the eigenvalues of the scattering matrix can turn into infinity at some real frequency. Simultaneously, the second eigenvalue vanishes at this frequency because $s^{(i)} = 1/(s^{(j)})^*$, $i \neq j$. A situation in which an eigenvalue of the scattering matrix vanishes at a real frequency corresponds to coherent perfect absorption (CPA) [58]. In this case, plane waves incident on the system do not produce scattered waves: $\phi_n^{(i)} \neq 0$, and $\psi_n^{(i)} = 0$. Therefore, at the beginning of lasing, i.e., when lasing power is still zero, a PT-symmetric optical system is at the same time a coherent perfect absorber. In the literature, such systems were called 'CPA lasers' [53, 54].

In Section 4.3, we discussed the possibility of observing a phase transition in PT-nonsymmetric systems that can be reduced to PT-symmetric ones after formally renaming the fields. In [59], a PT-nonsymmetric optical system consisting of two coherently coupled low-Q laser cavities was proposed. If a gain can be achieved in the medium in one of the cavities pumped by external radiation, then, beginning from the pump threshold, lasing appears in the system. In this case, the field distribution in the laser mode is asymmetric: the integrated field intensity in the pumped cavity is greater than that in the unpumped cavity [59]. If in addition the second cavity is also pumped, lasing is quenched at a critical value of the pump intensity in the second cavity. As the pump intensity in the second cavity is increased further, lasing again appears in the system. In this case, the field distribution in the laser mode is symmetric: the integrated field intensities in the cavities are equal [59].

Thus, in the system of two coupled cavities, we can observe the effect of laser quenching upon increasing the pump intensity, which is apparently counterintuitive, but is caused by the spatial inhomogeneity of pump radiation. It is important that this effect appears due to the phase transition in the system and is not related to the medium nonlinearity.

Indeed, we consider a system consisting of two layers with permittivities $\varepsilon_1 = 1.1 - 0.3\mathrm{i}$ and $\varepsilon_2 = 1.1 - 0.3\mathrm{i}\alpha$, where $\alpha \in [0, 1]$. The minus sign in the imaginary part of the permittivity corresponds to a gain medium. To determine the parameters of the system at which lasing occurs, it suffices to observe the position of the poles of the scattering matrix s in the complex frequency plane. If at least one of the poles of the scattering matrix lies in the upper frequency half-plane, the system produces lasing [46, 56, 57]. To determine the boundaries of the lasing region, it is very important to take the permittivity dispersion into account, because the optical thickness of the system increases as the radiation frequency increases and lasing conditions are satisfied at lower gains. As a result, lasing at high frequencies is observed at any gain. The consideration of dispersion leads to a decrease in the imaginary part of the permittivity with increasing frequency (the imaginary part decreases faster than $1/\mathrm{Re}\,\omega$), which allows rejecting nonphysical high-frequency lasing.

For simplicity and clarity, we first consider the system described above with the permittivity dispersion neglected and the poles at high frequencies not regarded as nonphysical.

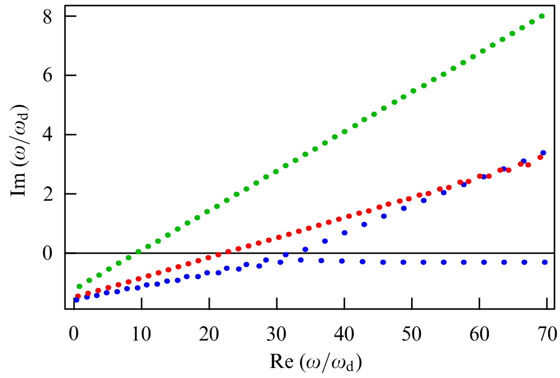


Figure 9. (Color online.) Position of poles of the scattering matrix in the complex frequency plane. The real part of the frequency is plotted on the abscissa and the imaginary part of the frequency $\mathrm{Im}\,(\omega/\omega_d)$, where $\omega_d = \pi c/(2d)$ on the ordinate axis. Green, red, and blue colors respectively indicate system poles with $\alpha = 1.0$, 0.5, and 0.

We consider three systems with $\alpha_1 = 0.0$, $\alpha_2 = 1.0$, and $\alpha_3 = 0.5$. In the first case, the coefficient α in the second layer is zero. In the second case, the pump is homogeneous over the system. The third case is intermediate between the first and the second.

As the radiation frequency increases, the poles of the system in all three cases move from the lower half-plane to the upper half-plane, and lasing begins (Fig. 9). At low frequencies, the poles of all three systems lie on straight lines in complex frequency planes and the slope of the straight lines increases with increasing gain in the second layer. At some frequency ($\omega/\omega_d \approx 30$), the eigenmodes in the first system are rearranged. As a result, the straight line in which poles are located splits into two straight lines with larger and smaller slopes than those of the initial line. The slope of one of these lines is even greater than that of the line in which the poles of the third system lie. As a result, a frequency exists at which the poles of the system without gain in the second layer are higher than the poles of the gain system in the second layer ($\omega/\omega_d \approx 65$). Thus, adding gain to the second layer can result in the lowering of a pole to the lower half-plane and laser quenching, as is described in [59]. We neglected the nonlinearity of the medium in our consideration, and therefore lasing quenching with increasing pumping is a purely linear effect.

6. 'Quasi-diode' PT-symmetric optical systems. The reciprocity principle

At present, the possibility of creating an optical computer that would use photons for data transfer, storage, and processing is actively being discussed [60, 61]. The elemental base of optical computers would be optical diodes transmitting a signal in only one direction.

The authors of [62] proposed an optical diode system based on a linear PT-symmetric system representing a two-mode waveguide in the form of a silicon rod with impurities periodically doped on its upper surface (Fig. 10a) [62]. The waveguide thickness was chosen such that it could support only two guided modes: an even mode with a maximum at the center of the rod cross section and an odd mode vanishing at the center of the rod cross section (Fig. 10b). Silicon impurities doped on one side of the waveguide neither amplify nor absorb light at the relevant frequencies. The

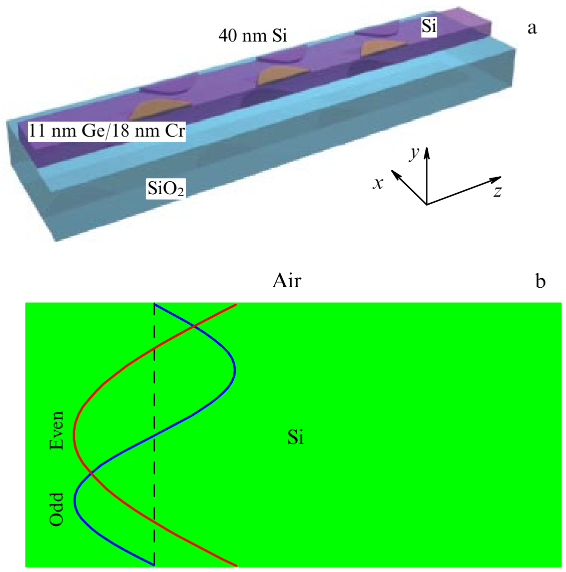


Figure 10. (Color online.) (a) Diagram of a PT-symmetric optical system of two coupled rectangular waveguides PT-symmetric along the z axis. (b) Field distributions in the even (red curve) and odd (blue curve) modes of the PT-symmetric optical system shown in Fig. 10a [62].

other side of the waveguide is doped with heterogeneous germanium/chromium impurities, which become amplifying elements when current passes through them. The location of impurities in the system can be described in terms of the permittivity distribution in the form [62]

$$\varepsilon(x, y, z) = \varepsilon_0(x, y) + \delta\varepsilon(x, y) \exp(iqz) , \tag{84}$$

where z is the waveguide axis and xy is the waveguide cross sectional area.

The integrated output field intensity of such a device depends on the side from which the signal is fed to the system [62]. Such a behavior of the system was related to its nonreciprocity, and it was proposed to use it as an optical diode.

In reality, the difference in the intensities is caused by different mode compositions of the output radiation and can be observed in reciprocal systems. Indeed, upon the incidence of the even mode, the transmission coefficient for this mode is independent of its incidence direction, in accordance with the reciprocity of the system, but the output amplitude of the odd mode depends on the side from which the even mode is incident. Notably, in [62], when the even mode is incident from one side, the output amplitude of the odd mode is zero (Fig. 11a), whereas for incidence from the other side, this amplitude is nonzero (Fig. 11b). Such a behavior does not contradict the system reciprocity and can also be observed in PT-nonsymmetric systems, as is seen in Fig. 11.

We formulate conditions under which the system is an optical diode. In linear electrodynamics, the relation between the amplitudes of incident and scattered waves is expressed in the matrix form

$$b_\mu = s_{\mu\nu} a_\nu , \tag{85}$$

where a_ν is the column of amplitudes of waves incident on the system, b_μ is the column of amplitudes of waves reflected from the system, and $s_{\mu\nu}$ is the scattering matrix. The subscripts μ and ν range from 1 to N, where N is the number of wave types (channels) that can be scattered from the system.

The optical system under study in the matrix formulation is equivalent to a network with four inputs. We take the even mode incident on the system from the left as the first mode, the odd mode incident from the left as the second mode, the even mode incident from the right as the third mode, and the odd mode incident from the right as the fourth mode (Fig. 12a).

The system is connected to the optical setup of a computer such that the signal is fed to one of the inputs of the network, for example, the first input (the even mode incident from the left) and is taken from the other, for example, the fourth input (the odd mode escaping from the right) (Fig. 12b). Upon propagation in the opposite direction, the signal is fed to the fourth input (the odd mode incident from the right) and is taken from the first input (the even mode escaping from the left) (Fig. 12c). For the system to operate as a diode, the condition $s_{14} \neq s_{41}$ should hold. In the general case, the scattering matrix of the optical diode should be nonsymmetric:

$$s_{\mu\nu} \neq s_{\nu\mu} . \tag{86}$$

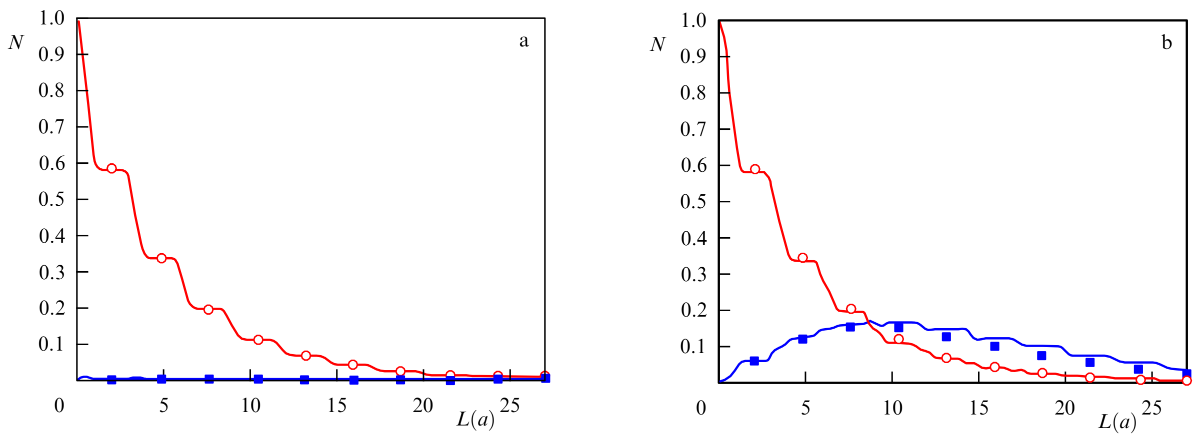


Figure 11. (Color online.) (a) Amplitudes of the even (red) and odd (blue) modes upon the incidence of the even mode from the left on the system, as a function of the system thickness. (b) Amplitudes of the even (red) and odd (blue) modes upon the incidence of the even mode from the right on the system, as a function of the system thickness. (From Supporting Online Material in [63].)

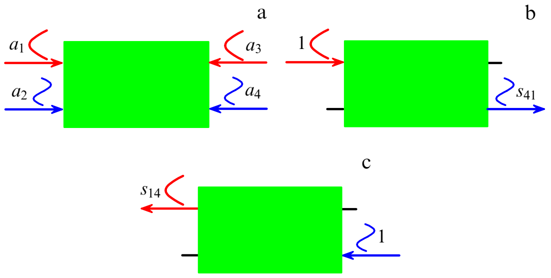


Figure 12. (a) Schematic representation of the scattering of waves from an optical system considered in [62]: a_1 and a_3 are the complex amplitudes of even modes incident on the system, a_2 and a_4 are the complex amplitudes of odd incident modes, b_1 and b_3 are the complex amplitudes of even modes scattered by the system, b_2 and b_4 are the complex amplitudes of odd modes scattered by the system. (b) Schematic representation of the incidence of the even mode with the unit amplitude from the left on the system. (c) Schematic representation of the incidence of the odd mode with the unit amplitude from the right on the system.

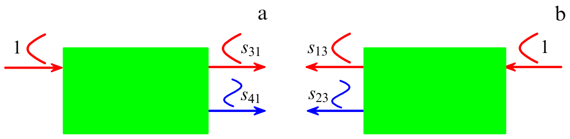


Figure 13. Schematic representation of the incidence of the even mode with the unit amplitude on the system (a) from the left and (b) from the right.

Based on the Lorentz lemma, it was shown in [64] that the scattering matrix of all linear optical systems is symmetric, $s_{\mu\nu} = s_{\nu\mu}$, if the permittivity and magnetic permeability are explicitly independent of time and are scalars or symmetric tensors,[10] $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$, $\mu_{\mu\nu} = \mu_{\nu\mu}$ (see Appendix 5). Such optical systems are called reciprocal. As follows from the above [see (86)], reciprocal optical systems cannot operate as an optical diode.

In [62], the first and second inputs, and also the third and fourth inputs, were combined, and therefore the output intensity was measured simultaneously for two inputs. In particular, upon the incidence of the even wave from the left (the first input), the output field intensity is the sum of intensities of the even and odd modes escaping from the right: $I \approx |s_{31}|^2 + |s_{41}|^2$ (Fig. 13a), while upon the incidence of the even wave from the right (the third input), the output field intensity is the sum of intensities of the even and odd modes escaping from the left: $I \approx |s_{13}|^2 + |s_{23}|^2$ (Fig. 13b). In this case, even in reciprocal systems, where $s_{31} = s_{13}$, the output integrated field intensity depends on the direction of the wave incident on the system because $s_{41} \neq s_{23}$ in general. The difference in the output powers at 'combined' inputs does not mean that the proposed PT-symmetric system is non-reciprocal, because this property is common for all reciprocal systems.

As an example illustrating these considerations, we discuss the system shown in Fig. 14a. This system is described by a scattering matrix similar to the scattering matrix of the system proposed in [62]. The difference is in the physical realization of scattering-matrix channels: in [62], these channels differ in symmetry, and a combined waveguide is excited through a common cross section, whereas channels in

[10] The permittivity and permeability tensors become nonsymmetric in the presence of an external field or nonzero magnetization [64].

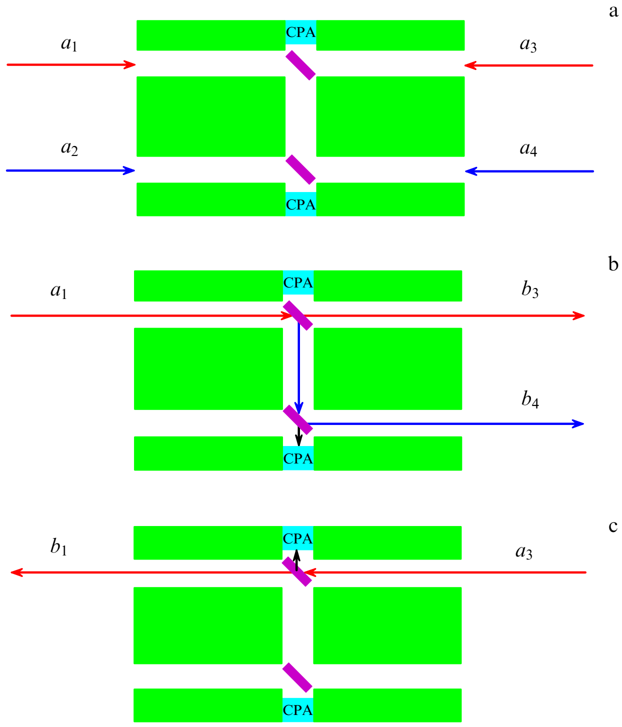


Figure 14. (a) Schematic representation of the scattering of waves from an optical system (notation as in Fig. 12a). (b) Schematic of the incidence of the even mode on the system from the left. (c) Schematic of the incidence of the even mode on the system from the right. The inclined rectangles are semi-transparent mirrors. CPA is a coherent perfect absorber. The situation is completely analogous to that for the scattering of waves by a PT-symmetric system.

our model have separate inputs (Fig. 14) and the mode symmetry plays no role at all. When an electromagnetic pulse is incident on the channel a_1 of the system (Fig. 14a), the amplitudes b_3 and b_4 of the output waves are nonzero (Fig. 14b). When an electromagnetic pulse is incident on the channel a_3, the output signal in the channel b_2 is absent (Fig. 14c). In other words, the system response is similar to the response of the system proposed in [62].

Thus, the PT-symmetric optical system proposed in [62] cannot operate as an optical diode, while the dependence of the output integrated field intensity on the direction of the input signal is not a specific feature of PT-symmetric systems only.

7. Conclusions

Interest in PT-symmetric systems appeared only recently. In 1998, Bender and Boettcher [28] showed that non-Hermitian Hamiltonians can have real eigenvalues if they commute with the product $\hat{P}\hat{T}$ of the spatial inversion and time reversal operators. Moreover, in systems with PT-symmetric Hamiltonians, a phase transition from real to complex eigenvalues of the Hamiltonian can be observed. This transition is accompanied by spontaneous PT-symmetry breaking for the Hamiltonian eigenfunctions. The mathematical apparatus of quantum mechanics with PT-symmetric Hamiltonians has been completely developed [31]. We note that quantum mechanics constructed in such a way is not an extension of ordinary quantum mechanics but represents a separate mathematical description. The difference between these quantum mechanics is in different definitions of the scalar product [32]. The PT-symmetric quantum mechanics remains a purely speculative construction, because quantum mechanical non-Hermitian PT-symmetric systems with a complex potential are absent in Nature. For this reason the study of PT-symmetric systems is currently concentrated on the fields of optics and the physics of semiconductors [65], where PT-symmetric systems can be constructed.

We have shown in this review that observing a phase transition in optical systems is extremely complicated. In particular, such a transition cannot be observed by varying only the frequency of an external field, while to observe the transition by varying pumping, the pump radiation should be spatially inhomogeneous. However, a 'hidden' phase transition can be observed in PT-nonsymmetric systems that can be reduced to PT-symmetric systems with the help of formal renaming and transformations.

We have shown that most of the 'unusual' properties assigned to PT-symmetric systems can also be observed in ordinary systems, including asymmetric refraction, quenching of lasing with increasing pump intensity [59], and 'nonreciprocity'.

We have discussed only linear PT-symmetric optical systems. The consideration of a broader range of effects (see, e.g., [66–71]) predicted for nonlinear PT-symmetric systems requires a separate paper.

Acknowledgments

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8. Appendices

A1 Nonorthogonality of eigenfunctions of a non-Hermitian Hamiltonian with real eigenvalues

Let all the eigenvalues E_k of a non-Hermitian Hamiltonian $\hat{H}$ with a discrete spectrum be real. In this case, it follows from the Schrödinger equation

$$\mathrm{i}\,\frac{\partial|\psi_k\rangle}{\partial t} = \hat{H}|\psi_k\rangle \qquad \text{(A1.1)}$$

that for an eigensolution $|\psi_k(t)\rangle = \exp(\mathrm{i}E_k t)|\psi_k(0)\rangle$ of the Hamiltonian, the wave-function norm $\langle\psi_k(t)|\psi_k(t)\rangle$ is preserved in time:

$$\langle\psi_k(t)|\psi_k(t)\rangle = \langle\psi_k(0)|\exp(-\mathrm{i}E_k t)\exp(\mathrm{i}E_k t)|\psi_k(0)\rangle = \langle\psi_k(0)|\psi_k(0)\rangle\,. \qquad \text{(A1.2)}$$

However, unlike the eigenfunctions of a Hermitian Hamiltonian, the eigenfunctions of $\hat{H}$ corresponding to different real eigenvalues are not orthogonal. To prove this, we multiply the Hermitian conjugate Schrödinger equation

$$-\mathrm{i}\,\frac{\partial\langle\psi_n|}{\partial t} = \langle\psi_n|\hat{H}^+ \qquad \text{(A1.3)}$$

by $|\psi_k\rangle$ from the right and Eqn (A1.1) by $\langle\psi_n|$ from the left, and subtract the second equation from the first:

$$\mathrm{i}\left(\langle\psi_n|\,\frac{\partial|\psi_k\rangle}{\partial t} + \frac{\partial\langle\psi_n|}{\partial t}\,|\psi_k\rangle\right) = \langle\psi_n|\hat{H} - \hat{H}^+|\psi_k\rangle = \mathrm{i}\,\frac{\partial\langle\psi_n|\psi_k\rangle}{\partial t}\,. \qquad \text{(A1.4)}$$

If the eigenfunctions of the Hamiltonian are orthogonal, $\langle\psi_n|\psi_k\rangle = 0$, then $\partial\langle\psi_n|\psi_k\rangle/\partial t = 0$. If the set of functions $|\psi_k\rangle$ is complete, it follows from (A1.4) that $\hat{H} = \hat{H}^+$, i.e., the Hamiltonian $\hat{H}$ is Hermitian. For a non-Hermitian Hamiltonian to have real eigenvalues, it is necessary that either $\langle\psi_n|\psi_k\rangle \neq 0$ for $n \neq k$ or the eigenfunctions of the Hamiltonian do not form a complete basis.

We here consider an important particular case of systems with PT-symmetric Hamiltonians ($\hat{P}\hat{T}\hat{H} = \hat{H}\hat{P}\hat{T}$). If all the eigenvalues of such a Hamiltonian are real, we can show that the system of its eigenfunctions is complete [32]. Therefore, the eigenfunctions of this Hamiltonian are nonorthogonal.

We note that it follows from (A1.2) that the norm is preserved only for the eigenfunctions of the Hamiltonian. It follows from the orthogonality of these functions that the norm of an arbitrary wave function $\psi = \sum_k c_k|\psi_k\rangle$ may not be preserved in time. Indeed,

$$\frac{\partial\langle\psi|\psi\rangle}{\partial t} = \frac{\partial\left(\sum_n c_n^*\langle\psi_n|\sum_k c_k|\psi_k\rangle\right)}{\partial t} = \frac{\partial\left(\sum_{k,n} c_n^* c_k\langle\psi_n|\psi_k\rangle\right)}{\partial t} = \sum_n |c_n|^2\,\frac{\partial\langle\psi_n|\psi_n\rangle}{\partial t} + \sum_{n\neq k} c_n^* c_k\,\frac{\partial\langle\psi_n|\psi_k\rangle}{\partial t} = \sum_{n\neq k} c_n^* c_k\,\frac{\partial\langle\psi_n|\psi_k\rangle}{\partial t} \neq 0\,. \qquad \text{(A1.5)}$$

For an arbitrary wave function, the quantity $\langle\psi|\hat{P}|\psi\rangle$ is preserved. Indeed, by multiplying (A1.1) from the left by $\langle\psi|\hat{P}$ and (A1.3) from the right by $\hat{P}|\psi\rangle$, and adding these expressions, we obtain

$$\mathrm{i}\,\frac{\partial\langle\psi|\hat{P}|\psi\rangle}{\partial t} = \langle\psi|\hat{H}^+\hat{P} - \hat{P}\hat{H}|\psi\rangle\,. \qquad \text{(A1.6)}$$

If $\hat{H}^+ = \hat{T}\hat{H}\hat{T}$ (see Appendix 2), then $\hat{H}^+\hat{P} - \hat{P}\hat{H} = \hat{T}\hat{H}\hat{T}\hat{P} - \hat{P}\hat{H} = \hat{H}\hat{T}\hat{P} - \hat{T}\hat{P}\hat{H} = \hat{H}\hat{P}\hat{T} - \hat{P}\hat{T}\hat{H} = 0$. Therefore, in this case, $\langle\psi|\hat{P}|\psi\rangle$ is a quantity preserved in time.

Thus, to construct closed quantum mechanics based on PT-symmetric Hamiltonians with real eigenvalues, it is necessary to redefine the scalar product and to take the vector $\langle\psi|\hat{P}$ as a bra vector. In this case, the rule for determining the means of physical quantities takes the form $\langle\hat{A}\rangle = \langle\psi|\hat{P}\hat{A}|\psi\rangle$.

A2 Pseudo-Hermitian systems

A necessary condition for the eigenvalues of a Hamiltonian to be real is its pseudo-Hermiticity:

$$\hat{H}^+ = \hat{S}^{-1}\hat{H}\hat{S}\,, \qquad \text{(A2.1)}$$

where $\hat{S}$ is any linear Hermitian operator. Notably, all PT-symmetric Hamiltonians with real eigenvalues are at the same time pseudo-Hermitian [29].

The pseudo-Hermiticity of the Hamiltonian is a necessary but not sufficient condition for the realness of the spectrum. In other words, pseudo-Hermitian Hamiltonians with complex eigenvalues exist. If a pseudo-Hermitian system with a Hamiltonian $\hat{H}(\mu)$ has purely real eigenvalues for some parameters μ and does not have real eigenvalues for other parameters, we are dealing with a phase transition from the states with real eigenvalues to the states with complex eigenvalues.

If the Hamiltonian of the system satisfies the condition $\hat{H}^+ = \hat{T}\hat{H}\hat{T}$,[11] then for $\hat{S} = \hat{P}$, the pseudo-Hermiticity condition (A2.1) coincides with the PT-symmetry condition (3):

$$\hat{H}^+ = \hat{T}\hat{H}\hat{T} = \hat{P}\hat{H}\hat{P} \Leftrightarrow \hat{P}\hat{T}\hat{H} = \hat{H}\hat{P}\hat{T}\,, \tag{A2.2}$$

where we used the fact that $\hat{P}^2 = \hat{1}$ and $\hat{T}^2 = \hat{1}$.

For arbitrary Hamiltonians ($\hat{H}^+ \neq \hat{T}\hat{H}\hat{T}$), the pseudo-Hermiticity condition (A2.1) with the operator $\hat{S} = \hat{P}$ does not coincide with PT-symmetry condition (3). For example, a physical system with the Hamiltonian

$$\hat{H} = \hat{p}^2 + \hat{x}^2\hat{p} \tag{A2.3}$$

is PT-symmetric, but not pseudo-Hermitian with the operator $\hat{S} = \hat{P}$ [29], and vice versa, a system with the Hamiltonian

$$\hat{H} = \hat{p}^2 + \mathrm{i}(\hat{x}^2\hat{p} + \hat{p}\hat{x}^2) \tag{A2.4}$$

is pseudo-Hermitian with the operator $\hat{S} = \hat{P}$, but not PT-symmetric [29].

For optical PT-symmetric systems, the Helmholtz equation can be reduced to the Schrödinger equation with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)\,, \tag{A2.5}$$

where $\hat{p} = \partial/\partial x + \ldots$, $V(x) = (\omega/c)^2\varepsilon(x)$, and $-\hbar^2/(2m) \rightarrow 1$. Because this Hamiltonian satisfies the condition $\hat{H}^+ = \hat{T}\hat{H}\hat{T}$, the PT-symmetric systems in optics are a particular case of pseudo-Hermitian systems.

A3 Restrictions imposed by the causality principle. Kramers–Kronig relations and the possibility of the existence of pseudo-Hermitian optical systems in a finite frequency range

The concept of the pseudo-Hermiticity of a system can be introduced in optics as was done for PT-symmetry condition (3). In one-dimensional and two-dimensional cases, Maxwell's equation reduce to the scalar Helmholtz equation [40]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 \varepsilon(x,z)\right) E(x,z) = 0\,, \tag{A3.1a}$$

[11] An example of such a Hamiltonian is given by Hamiltonians with a time-independent potential $V(x)$, not containing terms with the product of the momentum by the coordinate ($\hat{p}\hat{x}, \hat{p}^2\hat{x}, \hat{p}\hat{x}^2, \ldots$). Such Hamiltonians also include the simplest Hamiltonian $\hat{H} = \hat{p}^2/(2m) + V(\hat{x})$.

which formally coincides with the stationary Schrödinger equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\psi_k(x,z) - \frac{2m\big(V(x,z) - E_k\big)}{\hbar^2}\,\psi_k(x,z) = 0\,, \tag{A3.1b}$$

if we make the substitution $V(x,z) - E_k \rightarrow (\omega/c)^2\varepsilon(x,z)$, $\psi_k(x,z) = E(x,z)$, and $-\hbar^2/(2m) \rightarrow 1$.

The pseudo-Hermiticity condition (A2.1) in quantum optics for a system described by Schrödinger equation (A3.1b) amounts to the requirement $V^*(x,z) = \hat{S}V(x,z)\hat{S}$ for the potential energy. By analogy between the potential energy in quantum mechanics and the permittivity in optics, the pseudo-Hermiticity condition of an optical system can be defined as the condition imposed on the permittivity:

$$\operatorname{Re}\varepsilon(\omega,\mathbf{r}) = \hat{S}\big(\operatorname{Re}\varepsilon(\omega,\mathbf{r})\big)\hat{S}\,, \tag{A3.2a}$$

$$\operatorname{Im}\varepsilon(\omega,\mathbf{r}) = -\hat{S}\big(\operatorname{Im}\varepsilon(\omega,\mathbf{r})\big)\hat{S}\,. \tag{A3.2b}$$

We can see from the definition of pseudo-Hermiticity in (A3.2) that any pseudo-Hermitian optical system consists of amplifying and absorbing media. At the same time, the frequency dispersion of the permittivity plays a significant role in optical systems with amplification and absorption, and its consideration in gain media is fundamentally important. Indeed, if the dispersion of the gain medium is neglected, lasing would be observed in any arbitrarily small structure at the high frequencies at which the optical thickness is much greater [46, 57] than at low frequencies.

The permittivity is a function of the response of the medium to incident radiation, and as any response function, it must be causal (analytic in the upper complex frequency half-plane). The requirement that the permittivity be an analytic function in the upper complex frequency half-plane means that the real and imaginary parts of the permittivity are connected by the Kramers–Kronig relations [42]

$$\operatorname{Re}\varepsilon(\omega,\mathbf{r}) = \varepsilon_0 + \frac{1}{\pi}\,\text{v.p.}\int_{-\infty}^{\infty}\frac{\operatorname{Im}\varepsilon(\omega',\mathbf{r})}{\omega' - \omega}\,\mathrm{d}\omega'\,, \tag{A3.3a}$$

$$\operatorname{Im}\varepsilon(\omega,\mathbf{r}) = -\frac{1}{\pi}\,\text{v.p.}\int_{-\infty}^{\infty}\frac{\operatorname{Re}\varepsilon(\omega',\mathbf{r}) - \varepsilon_0}{\omega' - \omega}\,\mathrm{d}\omega'\,, \tag{A3.3b}$$

where ε_0 is the permittivity of the vacuum. By using Kramers–Kronig relations, we can show that pseudo-Hermiticity condition (A3.2) is satisfied only for a discrete frequency set.

Indeed, if the pseudo-Hermiticity condition in Eqn (A3.2b) holds for the imaginary part of the permittivity at any real frequencies, then

$$\hat{S}\big(\operatorname{Re}\varepsilon(\omega,\mathbf{r})\big)\hat{S} = \varepsilon_0 + \frac{1}{\pi}\,\text{v.p.}\int_{-\infty}^{\infty}\frac{\hat{S}\big(\operatorname{Im}\varepsilon(\omega',\mathbf{r})\big)\hat{S}}{\omega' - \omega}\,\mathrm{d}\omega'$$
$$= \varepsilon_0 - \frac{1}{\pi}\,\text{v.p.}\int_{-\infty}^{\infty}\frac{\operatorname{Im}\varepsilon(\omega',\mathbf{r})}{\omega' - \omega}\,\mathrm{d}\omega'\,. \tag{A3.4}$$

For the pseudo-Hermiticity condition to be satisfied for the real part of the permittivity, the equality $\operatorname{Re}\varepsilon(\omega,\mathbf{r}) = \hat{S}(\operatorname{Re}\varepsilon(\omega,\mathbf{r}))\hat{S}$ should hold. Comparing the expressions for $\operatorname{Re}\varepsilon(\omega,\mathbf{r})$ in (A3.3a) and $\hat{S}(\operatorname{Re}\varepsilon(\omega,-\mathbf{r}))\hat{S}$ in (A3.4), we see that they differ only in the sign at the integral. Therefore, the system pseudo-Hermiticity condition for the real part of the permittivity at all real frequencies can hold only when the identity

$$\frac{1}{\pi}\,\mathrm{v.p.}\int_{-\infty}^{\infty}\frac{\operatorname{Im}\varepsilon(\omega',\mathbf{r})}{\omega'-\omega}\,\mathrm{d}\omega'\equiv 0 \tag{A3.5}$$

holds, which is possible only for a completely transparent system ($\operatorname{Im}\varepsilon(\omega,\mathbf{r})\equiv 0$).

Following the logic of Section 3.2, we can easily show that the system pseudo-Hermiticity condition cannot be satisfied in any finite frequency range either. Hence, the system pseudo-Hermiticity condition can be satisfied only for a discrete frequency set.

A4 Orthogonality of the eigenvectors of the scattering matrix for a reciprocal system

Let ϕ_k^i be the ith eigenvector of the scattering matrix s_{nk} with the eigenvalue s^i:

$$s_{nk}\phi_k^{(i)} = s^{(i)}\phi_n^{(i)}\,, \tag{A4.1}$$

where summation over the superscripts in parentheses is not performed. Equation (A4.1) can be written in the form

$$\left(\phi_k^{(j)}\right)^{\mathrm{T}} s_{kn} = s^{(j)}\left(\phi_n^{(j)}\right)^{\mathrm{T}}. \tag{A4.2}$$

Multiplying both sides of Eqn (A4.1) by $\left(\phi_n^{(j)}\right)^{\mathrm{T}}$ and both sides of (A4.2) by $\phi_n^{(i)}$, and subtracting the first equation from the second, we obtain

$$\left(s^{(i)}-s^{(j)}\right)\left(\phi_n^{(j)}\right)^{\mathrm{T}}\phi_n^{(i)} = (\phi_n^j)^{\mathrm{T}}(s_{nk}-s_{kn})\,\phi_k^i\,. \tag{A4.3}$$

The scattering matrix of a reciprocal medium is symmetric, $s_{nk}=s_{kn}$ [64]. Therefore, if the eigenvalues are different, the eigenvectors of the scattering matrix are orthogonal:

$$(\phi_n^j)^{\mathrm{T}}\phi_n^i = 0\,. \tag{A4.4}$$

In the case of one-dimensional PT-symmetric optical systems, the scattering matrix eigenvalues coincide with each other only at the point of a phase transition (see Section 5.1 and [53]) from PT-symmetric eigenvectors to PT-nonsymmetric eigenvectors. In this case, the choice of eigenvectors is arbitrary and they can always be chosen to be mutually orthogonal [39].

A5 Properties of the *c* matrix

We consider some results that are important in Section 5.1. It follows from (76) that the c^{ij} matrix is a matrix representation of the operator $\hat{P}\hat{T}$ in the basis of eigenfunctions $\phi_k^{(i)}$ of the scattering matrix,

$$\hat{P}\hat{T}\phi_k^i = c^{ij}\phi_k^j\,. \tag{A5.1}$$

Therefore, the c^{ij} matrix has the same properties as the operator $\hat{P}\hat{T}$. For example, it follows from the relations $(\hat{P}\hat{T})^2=\hat{1}$ and $(\hat{P}\hat{T})^+=\hat{P}\hat{T}$ that $c^{ij}c^{jl}=\delta^{il}$ and $(c^{ij})^*=c^{ji}$. It follows from the Hermiticity of the c^{ij} matrix that c^{11} and c^{22} are real and $c^{12}=(c^{21})^*$. From the matrix equation $c^{ij}c^{jl}=\delta^{il}$, we obtain

$$(c^{11})^2+c^{12}c^{21}=1\,, \tag{A5.2a}$$

$$c^{12}(c^{11}+c^{22})=0\,, \tag{A5.2b}$$

$$c^{21}(c^{11}+c^{22})=0\,, \tag{A5.2c}$$

$$(c^{22})^2+c^{12}c^{21}=1\,. \tag{A5.2d}$$

We can see from (A5.2a) and (A5.2b) that $c^{11}=\pm c^{22}$.

If $c^{11}=c^{22}$, then it follows from (A5.2b) and (A5.2c) that $c^{12}=c^{21}=0$. In this case, $c^{11}=c^{22}=\pm 1$ and

$$c=\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{A5.3}$$

If $c^{11}=-c^{22}$, then Eqns (A5.2b) and (A5.2c) do not impose additional restrictions on c^{12} and c^{21}. In this case, it follows from the condition $c^{12}=(c^{21})^*$ and Eqn (A5.2) that

$$(c^{11})^2+|c^{12}|^2=1\,, \tag{A5.4a}$$

$$(c^{11})^2+|c^{21}|^2=1\,. \tag{A5.4b}$$

Because $c^{11}\in\mathbb{R}$ and $c^{11}=-c^{22}$, the c matrix can be written as

$$c=\begin{pmatrix}\cos\theta & \sin\theta\exp(\mathrm{i}\phi)\\ \sin\theta\exp(-\mathrm{i}\phi) & -\cos\theta\end{pmatrix}, \tag{A5.5}$$

where θ and ϕ are arbitrary real numbers.

A6 Symmetry of the scattering matrix of an optical system as a corollary of the Lorentz lemma

For linear optical systems with the permittivity and magnetic permeability that are independent of time and are scalars or symmetric tensors ($\varepsilon_{\mu\nu}=\varepsilon_{\nu\mu}$, $\mu_{\mu\nu}=\mu_{\nu\mu}$), the Lorentz lemma holds [64], which relates the electric and magnetic fields $\mathbf{E}_i$ and $\mathbf{H}_i$ produced by electric and magnetic currents j_i^{e} and j_i^{m}, $i=\{1;2\}$,

$$\int_S\left(\mathbf{E}_1\times\mathbf{H}_2-\mathbf{E}_2\times\mathbf{H}_1\right)\mathrm{d}S = \int_V\left(j_1^{\mathrm{m}}H_2-j_2^{\mathrm{m}}H_1+j_1^{\mathrm{e}}E_2-j_2^{\mathrm{e}}E_1\right)\mathrm{d}V\,. \tag{A6.1}$$

It follows from the Lorentz lemma that the scattering matrix of the system is symmetric:

$$s_{\mu\nu}=s_{\nu\mu}\,. \tag{A6.2}$$

We prove this for a multimode waveguide with two open ends, which we conventionally call the left and right ends. We assume that the distribution of $\mathbf{E}_1$ and $\mathbf{H}_1$ in the cross section of the left end coincides with one of the waveguide eigenmodes μ and corresponds to propagation from left to right, $\mathbf{E}_1^{\mathrm{l}}=\mathbf{E}_\mu^{(+)}$, $\mathbf{H}_1^{\mathrm{l}}=\mathbf{H}_\mu^{(+)}$, and in the cross section of the right end of the waveguide, is equal to the sum over all waveguide eigenmodes with unknown coefficients: $\mathbf{E}_1^{\mathrm{r}}=\sum_{\kappa=1}^N s_{\mu\kappa}\mathbf{E}_\kappa^{(+)}$, $\mathbf{H}_1^{\mathrm{r}}=\sum_{\kappa=1}^N s_{\mu\kappa}\mathbf{H}_\kappa^{(+)}$. We also assume that the distribution of fields $\mathbf{E}_2$ and $\mathbf{H}_2$ in the cross section of the right end of the waveguide coincides with one of the waveguide eigenmodes ν and corresponds to propagation from right to left, $\mathbf{E}_2^{\mathrm{r}}=\mathbf{E}_\nu^{(-)}$, $\mathbf{H}_2^{\mathrm{r}}=\mathbf{H}_\nu^{(-)}$, and in the cross section of the left end of the waveguide, is equal to the sum over all waveguide eigenmodes with unknown coefficients: $\mathbf{E}_2^{\mathrm{l}}=\sum_{\kappa=1}^N s_{\nu\kappa}\mathbf{E}_\kappa^{(-)}$, $\mathbf{H}_2^{\mathrm{l}}=\sum_{\kappa=1}^N s_{\nu\kappa}\mathbf{H}_\kappa^{(-)}$.[12]

We choose the integration volume in (A6.1) such that the waveguide lies inside this volume and the integration surface intersects the waveguide only over the cross sections of the

[12] For modes with identical indices but different propagation directions, the condition $\mathbf{E}_\mu^{(+)}\times\mathbf{H}_\mu^{(+)}=-\mathbf{E}_\mu^{(-)}\times\mathbf{H}_\mu^{(-)}$ holds.

open ends. We also assume that the integration volume does not contain external electric and magnetic currents. In this case, expression (A6.1) transforms into

$$\int_{S^{\mathrm{l}}} \left(\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1\right) \mathrm{d}S = \int_{S^{\mathrm{r}}} \left(\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1\right) \mathrm{d}S\,, \qquad (\mathrm{A}6.3)$$

where S^{l} and S^{r} are the cross-sectional surfaces for the left and right ends of the waveguide. Substituting the expressions for $\mathbf{E}_i$ and $\mathbf{H}_i$ in the left and right waveguide cross sections, we obtain

$$\int_{S^{\mathrm{l}}} \left[\mathbf{E}_{\mu}^{(+)} \times \left(\sum_{\kappa=1}^{N} s_{\nu\kappa} \mathbf{H}_{\kappa}^{(-)}\right) - \left(\sum_{\kappa=1}^{N} s_{\nu\kappa} \mathbf{E}_{\kappa}^{(-)}\right) \times \mathbf{H}_{\mu}^{(+)}\right] \mathrm{d}S = \int_{S^{\mathrm{r}}} \left[\left(\sum_{\kappa=1}^{N} s_{\mu\kappa} \mathbf{E}_{\kappa}^{(+)}\right) \times \mathbf{H}_{\nu}^{(-)} - \mathbf{E}_{\nu}^{(-)} \times \left(\sum_{\kappa=1}^{N} s_{\mu\kappa} \mathbf{H}_{\kappa}^{(+)}\right)\right] \mathrm{d}S\,. \qquad (\mathrm{A}6.4)$$

Pulling the summations outside the parentheses, we have

$$\sum_{\kappa=1}^{N} s_{\nu\kappa} \int_{S^{\mathrm{l}}} \left(\mathbf{E}_{\mu}^{(+)} \times \mathbf{H}_{\kappa}^{(-)} - \mathbf{E}_{\kappa}^{(-)} \times \mathbf{H}_{\mu}^{(+)}\right) \mathrm{d}S = \sum_{\kappa=1}^{N} s_{\mu\kappa} \int_{S^{\mathrm{r}}} \left(\mathbf{E}_{\kappa}^{(+)} \times \mathbf{H}_{\nu}^{(-)} - \mathbf{E}_{\nu}^{(-)} \times \mathbf{H}_{\kappa}^{(+)}\right) \mathrm{d}S\,. \qquad (\mathrm{A}6.5)$$

Using the fact that the amplitudes of eigenmodes in a waveguide with a constant cross section can be normalized such that the condition

$$\int_{S^{\mathrm{l}},\, S^{\mathrm{r}}} \left(\mathbf{E}_{\mu}^{(+)} \times \mathbf{H}_{\nu}^{(-)} - \mathbf{E}_{\nu}^{(-)} \times \mathbf{H}_{\mu}^{(+)}\right) \mathrm{d}S = \delta_{\mu\nu}$$

holds [64], we obtain

$$s_{\nu\mu} \int_{S^{\mathrm{l}}} \left(\mathbf{E}_{\mu}^{(+)} \times \mathbf{H}_{\mu}^{(-)} - \mathbf{E}_{\mu}^{(-)} \times \mathbf{H}_{\mu}^{(+)}\right) \mathrm{d}S = s_{\mu\nu} \int_{S^{\mathrm{r}}} \left(\mathbf{E}_{\nu}^{(+)} \times \mathbf{H}_{\nu}^{(-)} - \mathbf{E}_{\nu}^{(-)} \times \mathbf{H}_{\nu}^{(+)}\right) \mathrm{d}S\,, \qquad (\mathrm{A}6.6)$$

$$s_{\nu\mu} = s_{\mu\nu}\,. \qquad (\mathrm{A}6.7)$$

Because equality (A6.7) is valid for any μ and ν, the scattering matrix of the system is symmetric under the index permutation.

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