Delay-time statistics for diffuse waves

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(Received 28 January 1999)

We formulate a theory for the statistics of the dynamics of a classical wave propagating in random media by analyzing the frequency derivative of the phase under the assumption of a Gaussian process. We calculate frequency correlations and probability distribution functions of dynamical quantities, as well the first non-Gaussian C_2 correction. In A. Z. Genack, P. Sebbah, M. Stoytchev, and B. A. van Tiggelen, Phys. Rev. Lett. **82**, 715 (1999), microwave measurements have been performed to which this theory applies.
[S1063-651X(99)04506-7]

PACS number(s): 42.25.−p

I. SCOPE

Fluctuations in wave propagation in disordered media are extensively treated in standard references [2,3]. Perhaps the best known feature is the apparently Gaussian statistics of the complex wave field, with real and imaginary parts as independent variables. This results in the familiar Rayleigh statistics $P(I)\sim\exp(-I/\langle I\rangle)$ of the intensity I. The Gaussian process is a natural consequence of the central limit theorem if one assumes that the complex field results from a coherent superposition of many independent wave trajectories.

Modern speckle theory, founded in the 1980s and still developing rapidly, changes this simple picture considerably [3,4]. New features in speckle statistics have been observed such as short- and long-range intensity correlations, e.g., in frequency [5] or time [6], the memory effect [7], universal fluctuations [8,9], and non-Rayleigh statistics [10] for the intensity, many of these explained in terms of non-Gaussian field statistics. An old tool from nuclear physics, random matrix theory (RMT), has been successfully introduced in this field and seems to capture many ''universal'' aspects [11]. Only recently, RMT has solved the fundamental problem of the statistical properties of the so-called phase delay times—which for classical waves are the derivatives of the phase shifts of the S matrix with respect to frequency $d\phi/d\omega$—at least for chaotic billiards where Dyson circular ensembles are known to apply [12,13]. The applicability of the Dyson RMT in disordered systems has been estimated to hold for energies small compared to the Thouless energy [14]. Pioneering work by Dorokhov [15] and Mello, Pereyra, and Kumar [16] extended RMT to disordered wires. The extension of DMPK theory towards dynamical problems is under active investigation [17,18].

The popularity of the phase derivative $d\phi/d\omega$ stems from its interpretation as a time delay in scattering [19]. This quantity describes the dynamics of a very narrow-band wave packet. In homogeneous media it relates directly to the group velocity v_G [20] whereas in random media its ensemble average is inversely proportional the transport velocity v_E figuring in the diffusion constant of the average intensity [22]. Unlike v_E, however, $d\phi/d\omega$ can be considered for arbitrary realizations. The restriction to narrow-band wave packets introduces sometimes apparently interpretational problems, such as negative group velocities or negative delay times, that have been extensively studied in literature [21].

The phase delay time relates directly to a fundamental dynamic quantity in condensed matter, namely the number of states per frequency interval $d\omega$ inside the scattering medium $N(\omega)$ [23],

$$\frac{1}{\pi}\sum_{a,b}^{2M} I_{ab}\frac{d\phi_{ab}}{d\omega}=N(\omega), \tag{1}$$

where $\sqrt{I_{ab}}\exp(i\phi_{ab})\equiv t_{ab}$ is the complex transition amplitude from mode a to b. Unlike the summation in the Landauer formula for conductance, the summation runs over the M channels in both reflection and transmission. Equation (1) is a manifestation of Friedel's theorem [24], originally devised for screening problems in the solid state, though with elegant applications to many scattering problems [25–27], including ones in RMT [28]. Alt'shuler and Shklovskii [14] demonstrated the central role of the statistics of $N(\omega)$ in the understanding of the relation between level repulsion—described as a concept in RMT—and universal conductance fluctuations, a basic element in modern speckle theory, a relation confirmed numerically [17]. It is crucial that the density of states in an open (scattering) medium is ill-defined due to the finite (Thouless) width of the levels. As shown in Ref. [23], the left-hand side of Eq. (1) is well-defined and proportional to the integral $\int d\mathbf{r}|\psi_\omega(\mathbf{r})|^2$ over the sample, which is—within the original Friedel argument—recognized as the ''stored charge.'' For light it equals the stored electromagnetic energy [27].

Though not free from controversy, Eq. (1) calls for the interpretation of $W_{ab}\equiv I_{ab}d\phi_{ab}/d\omega$ as the weighted delay time for a transition from channel a to b [29], to be distinguished from the ''proper'' delay times defined as the eigenvalues of the Wigner-Smith matrix $\mathbf{Q}=-i\mathbf{S}^*\cdot\partial\mathbf{S}/\partial\omega$, and its trace $\mathrm{Tr}\,\mathbf{Q}=\Sigma_{a,b}W_{ab}=\pi N(\omega)$, called the Heisenberg time τ_H. The channel average $\tau_H/2M$ is associated with *the* Wigner-Smith phase delay time τ_W. Finally, one could call $d\phi_{ab}/d\omega$ the single channel delay time for a transition from channel a to b, *irrespective* the transition probability I_{ab}.

The intention of this work is to formulate a statistical theory for the dynamical matrix elements $d\phi_{ab}/d\omega$ and W_{ab} for diffuse waves, using concepts developed for the static cross section I_{ab} [4]. This theory can be applied equally well to phenomena involving phase variations with other variables. The choice of frequency is stimulated by microwave experiments [1,30]. We shall adopt the ''C_1 approximation,'' which is known to work best for the static ''one channel in–one channel out'' matrix element I_{ab} (provided the conductance $g=M\ell/L\gg 1$), but which as far as we know has not been worked out for dynamic quantities such as $d\phi_{ab}/d\omega$ and W_{ab}. The summation $\Sigma_b W_{ab}$ equals the *diagonal* element Q_{aa} of the Wigner-Smith matrix and may, like the total transmission $\Sigma_b T_{ab}$ from channel a, be subject to C_2 correlations [5]. It is known that even a Gaussian process contains phase correlations between speckle spots [31]. The last section of this work will address the first results for C_2 frequency correlations in the dynamic matrix element W_{ab}. As yet, we have not been able to formulate a non-Gaussian theory for $d\phi_{ab}/d\omega$. We note, however, that experiments have shown this quantity to be highly Gaussian [1].

II. GAUSSIAN APPROXIMATION

The C_1 approximation is equivalent to the assumption of a circular complex Gaussian process [2] of the complex set t_{ab}. A circular process for K complex field amplitudes $E_i(b)=t_{ab}(i)E_i^{\rm in}(a)$ requires that $\langle E_i\rangle=0$ and that $\langle E_i E_j\rangle=0$. The index i here labels K different frequencies for a given channel transition ab. The joint distribution is given by

$$P(E_1,\ldots,E_K)=\frac{1}{\pi^K \det \mathbf{C}}\exp\left(-\sum_{i,j=1}^{K}\bar{E}_i C_{ij}^{-1}E_j\right), \quad (2)$$

where $C_{ij}=\langle E_i\bar{E}_j\rangle$ is the Hermitian variance matrix. We shall normalize $\langle E_j\bar{E}_j\rangle=1$ for all j, assuming $\langle I_j\rangle=1$ to be independent of j. For small frequency difference $\omega_1-\omega_2=\omega$, we can make the expansion $C_{12}=1+ia\omega+b\omega^2+O(\omega^3)$, where a and b can be calculated from diffusion theory [4,32], which involves the diffusion constant D, sample length L, absorption length L_a, and transport velocity v_E. The latter contains the Wigner delay time of the scattering objects [22] and thus forms the crucial link between microdynamics and macrodynamics.

Probability distributions can be derived for $K=2$ using a change of variables $E_j=A_j\exp(i\phi_j)$. As $\omega\to 0$, the stochastic variables can be chosen as $I=A^2$, $\phi'\equiv d\phi_{ab}/d\omega$, $R\equiv d\ln A_{ab}/d\omega$, and ϕ_{ab}. Integrating out the phase shift ϕ_{ab} yields for the joint distribution,

$$P(I,\phi',R)=\frac{I}{\pi Q a^2}\exp(-I)\times\exp\left[-\frac{I}{Qa^2}(\phi'-a)^2-\frac{I}{Qa^2}R^2\right]. \quad (3)$$

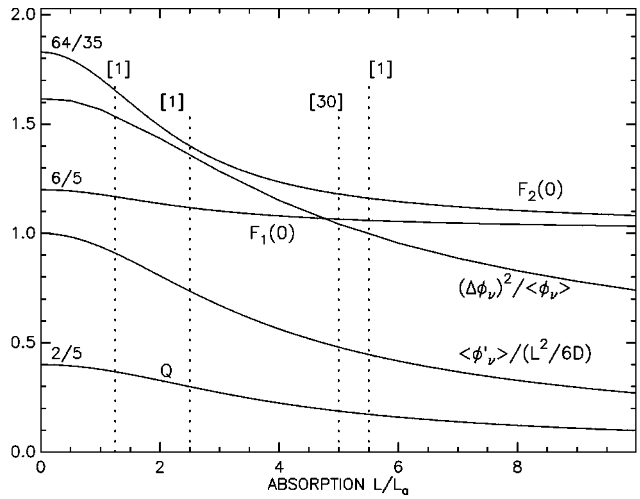

FIG. 1. Several parameters which appear in the statistics of the delay time are shown as functions of the absorption L/L_a in transmission through a thick slab of length L. Dashed vertical lines estimate their values in the experiments of Refs. [1] and [30]. The function $F_1(0)$ denotes the short range ''C_1'' contribution to $\langle W^2\rangle-\langle W\rangle^2$; $(1/g)F_2(0)$ is the non-Gaussian ''C_2'' contribution observed in Ref. [1]. Q is the dimensionless parameter determining the probability distribution of the quantities W and ϕ'; the average delay time $\langle\phi'_\omega\rangle$ has been normalized to the diffuse traversal time $L^2/6D$. The relation $(\Delta\phi_\nu)^2\sim\langle\phi_\nu\rangle$ for the cumulative phase at frequency ν is given by Eq. (15).

The distribution functionally depends on a single parameter $Q\equiv -2b/a^2-1>0$, which is shown in Fig. 1 as a function of absorption L/L_a. From the diffusion formula for $C(\Omega)$ in transmission used in Ref. [4], it follows that

$$Q=\frac{X^2-2\sinh^2 X+(\frac{1}{2})X\sinh 2X}{(X\cosh X-\sinh X)^2}, \quad (4)$$

with $X=L/L_a$.

In the absence of absorption, the mean delay time $\langle\phi'\rangle=a$ equals the diffuse traversal time $L^2/6D$ for waves in transmission, and $4L/3v_E$ in reflection. In Fig. 1 we see how $\langle\phi'\rangle$ decays as absorption comes in. Measuring $\langle\phi'\rangle$ in reflection and transmission would give access to both transport mean free path $\ell=3D/v_E$ and transport velocity v_E.

Even in the Gaussian approximation, I and ϕ' are correlated observables. For constant intensity I the delay time is normally distributed with spread $\Delta\phi'/\langle\phi'\rangle=\sqrt{Q/2I}$. This has been confirmed experimentally [1]. The phase in dark points becomes ill-defined [31], which causes strong fluctuations in ϕ' at low intensities. Upon integration over I and R one finds

$$P\left(\hat{\phi}'\equiv\frac{\phi'}{\langle\phi'\rangle}\right)=\frac{1}{2}\frac{Q}{[Q+(\hat{\phi}'-1)^2]^{3/2}}. \quad (5)$$

This algebraic law agrees with experimental data for microwaves in transmission [1] over *seven* orders of magnitude. It has the property that $\langle(\phi')^2\rangle=\infty$, though any finite frequency grid $\Delta\omega$ transforms this divergence into a finite value $-\ln\Delta\omega$. From Eq. (3), we find for the distribution of the dynamic matrix element W_{ab},

$$P\left(\hat{W}\equiv\frac{W}{\langle W\rangle}\right)=\frac{1}{\sqrt{Q+1}}\exp\left(\frac{-2|\hat{W}|}{\theta(\hat{W})+\sqrt{Q+1}}\right). \tag{6}$$

The Heaviside function $\theta(x)$ vanishes for $x<0$ and equals 1 for $x>0$. The average $\langle W\rangle=\langle I\rangle\langle\phi'\rangle\sim L/2Mv_E$ both in transmission and reflection. Like the intensity I, W has an exponential distribution but unlike I it can take negative values. Though less probable, the existence of negative "delay" times is an interesting feature that is also observed in experiments [1] and is allowed by scattering theory [19]. By Eq. (3) these are most probable in "dark spots." In transmission from a thick slab without absorption $Q=2/5$ and positive values for W are 12 times more probable than negative values. In reflection [22], $Q\approx(3L/7\ell)^2\gg 1$, implying nearly equal probabilities for positive and negative W_{ab}.

III. GAUSSIAN THEORY FOR FREQUENCY CORRELATIONS

Correlation functions at two close frequencies provide a sensitive test of the validity of the Gaussian approximation in the experiment. Frequency correlations of ϕ'_{ab} and W_{ab} can be obtained from Eq. (2) with $K=4$ at the frequencies $\nu\pm\omega/2\pm\Omega/2$ in the limit $\omega\to 0$. The correlation matrix we need to study is

$$\mathbf{C}(\omega,\Omega)=\begin{pmatrix}1 & C(\omega) & C(\Omega) & C(\Omega+\omega)\\ \overline{C}(\omega) & 1 & C(\Omega-\omega) & C(\Omega)\\ \overline{C}(\Omega) & \overline{C}(\Omega-\omega) & 1 & C(\omega)\\ \overline{C}(\Omega+\omega) & \overline{C}(\Omega) & \overline{C}(\omega) & 1\end{pmatrix}. \tag{7}$$

For $\omega=0$ this matrix contains a doubly degenerated eigenvalue $\lambda_{1,2}=0$ and two eigenvalues $\lambda_{3,4}(\Omega)=2\pm 2|C(\Omega)|$. For $\omega\to 0$ one gets

$$\frac{\lambda_{1,2}(\Omega)}{\omega^2}\equiv\xi_{1,2}(\Omega)=\frac{Qa^2}{2}+\frac{1}{2}\frac{|C'(\Omega)-iaC(\Omega)|^2}{1-|C(\Omega)|^2}\pm\frac{1}{2}\frac{|z(\Omega)|}{1-|C(\Omega)|^2}, \tag{8}$$

with $z(\Omega)\equiv\overline{C}''(\Omega)[1-|C(\Omega)|^2]-a^2\overline{C}(\Omega)+2ia\overline{C}'(\Omega)+C(\Omega)\overline{C}'(\Omega)^2$. The corresponding four eigenfunctions can be derived straightforwardly, among which the first two will be required to order ω, as customary in second-order perturbation theory. A careful analysis for $\omega\to 0$ in the subspace spanned by the first two eigenvectors shows that the joint distribution for the four complex fields can be transformed into

$$\begin{aligned}P(A_1,A_3,A_1',A_3',\phi_1,\phi_3,\phi_1',\phi_3')=&\frac{1}{\pi^4}\frac{A_1^2A_3^2}{\xi_1\xi_2\lambda_3\lambda_4}\exp-\frac{1}{4\xi_1}|A_1'-\overline{\mu^-}A_1+iA_1\phi_1'+(A_3'+\mu^-A_3+iA_3\phi_3')\exp(i\phi_{13}-i\rho)|^2\\ &\times\exp-\frac{1}{4\xi_2}|A_1'-\overline{\mu^+}A_1+iA_1\phi_1'-(A_3'+\mu^+A_3+iA_3\phi_3')\exp(i\phi_{13}-i\rho)|^2\\ &\times\exp-\frac{1}{\lambda_3}|A_1+A_3\exp(i\phi_{13}+i\tau)|^2\times\exp-\frac{1}{\lambda_4}|A_1-A_3\exp(i\phi_{13}+i\tau)|^2,\end{aligned} \tag{9}$$

where $\rho(\Omega)$ and $\tau(\Omega)$ are the complex phases of $z(\Omega)$ and $C(\Omega)$,

$$\mu^{\pm}(\Omega)\equiv\frac{\gamma_1(\Omega)\pm\gamma_2(\Omega)\exp[i\rho(\Omega)]}{1-|C(\Omega)|^2}, \tag{10}$$

and $\gamma_1(\Omega)\equiv C(\Omega)\overline{C'(\Omega)}+ia$, $\gamma_2(\Omega)\equiv iaC(\Omega)-C'(\Omega)$. Equation (9) implies many correlations between amplitudes, phase, and phase derivatives, only two of which we shall discuss. We remark that it is straightforward to integrate out the amplitude derivatives $A'_{1,3}$.

A. $Id\phi/d\omega$ frequency correlation

The calculation of the normalized frequency correlation function of $W_{ab}\equiv I_{ab}d\phi_{ab}/d\omega$ involves straightforward integrations of Eq. (9) that can all be done analytically,

$$\begin{aligned}&\langle\hat{W}_{ab}(\nu-\Omega/2)\hat{W}_{ab}(\nu+\Omega/2)\rangle_c\\ &=\frac{1}{2a^2}[|C'(\Omega)|^2-\mathrm{Re}\,C(\Omega)\overline{C}''(\Omega)]\\ &\equiv F_1(\Omega).\end{aligned} \tag{11}$$

This result can be obtained more easily using a method that will be introduced later to calculate long-range correlations, using the Gaussian rules to evaluate a correlation involving four fields.

The variance $(\Delta \hat{W}_{ab})^2 = F_1(0)$ has been plotted in Fig. 1 as a function of absorption and in transmission. Without absorption, the diffusion approximation for $C(\Omega)$ in transmission [4] predicts $F_1(0)=6/5$. For large Ω one can show that $F_1(\Omega) \to (\frac{1}{2}) \exp -(\sqrt{2\Omega L^2/D})$.

B. $d\phi/d\omega$ frequency correlation

The derivation of the frequency correlation function for the single channel delay time $\phi'_{ab}(\nu)$ involves more work. After analytically integrating out the amplitude derivatives $A'_{1,3}$, an integral over the phase shift ϕ_{13} remains that must be done numerically. Equation (9) finally yields

$$\left\langle \frac{d\phi}{d\omega}(\nu-\Omega/2)_{ab} \frac{d\phi}{d\omega}(\nu+\Omega/2)_{ab} \right\rangle \equiv \left\langle \frac{d\phi}{d\omega}(\omega) \right\rangle^2 [1 + C_\phi(\Omega)]$$
$$= \frac{1}{4\pi^2(1-|C|^2)} \int_{-2\pi}^{2\pi} d\phi (2\pi - |\phi|) \Bigg[-2\,\mathrm{Re}(z e^{-i\phi}) \mathcal{H}_0(R_\phi) + \frac{\mathcal{H}_2(R_\phi)(\mathrm{Im}^2\,\gamma_1 + \mathrm{Im}^2\,\gamma_2 e^{i\phi}) - 2\,\mathrm{Im}\,\gamma_1\,\mathrm{Im}\,\gamma_2\,\mathcal{H}_1(R_\phi)}{1-R_\phi^2} \Bigg], \tag{12}$$

where $R_\phi \equiv \mathrm{Re}\, C(\Omega) e^{i\phi}$. We have introduced the functions $\mathcal{H}_0(x) = \arctan[\sqrt{(1+x)/(1-x)}]/\sqrt{1-x^2}$, $\mathcal{H}_1 = 2\mathcal{H}_0 + x$, and $\mathcal{H}_2 = 2x\mathcal{H}_0 + 1$.

As $\Omega \to 0$, Eq. (12) predicts a logarithmic divergence, not encountered in intensity correlation functions. For large frequency shifts, we find that

$$C_\phi(\Omega) = |C(\Omega)|^2 \quad (|C| \ll 1), \tag{13}$$

i.e., C_ϕ becomes identical to the C_1 intensity correlation function which decays exponentially. In Ref. [1] the correlation function (12) has been compared to experiment, in a regime where the intensity is known to be subject to large non-Gaussian fluctuations. The excellent agreement between Eqs. (5) and (12) and experimental data seems to exclude the existence of non-Gaussian (long-range) correlations in the single channel phase delay time $d\phi_{ab}/d\omega$. At present we have no easy explanation for this phenomenon. In Fig. 2 we show the correlation function $C_\phi(\Omega)$ for different absorption lengths.

The Ω integral of the phase-delay correlation $C_\phi(\Omega)$ is required for the variance $\Delta\phi_\nu$ of the *cumulative* phase. The latter is defined as

$$\phi_{ab}(\nu) \equiv \int_{\nu_0}^{\nu} d\omega \frac{d\phi_{ab}}{d\omega}(\omega) \tag{14}$$

with respect to some reference frequency ν_0. The short-range frequency correlation of $d\phi/d\omega$ ensures that to a good approximation $\phi(\omega)$ is Gaussian distributed with a variance proportional to its average [30]. The relation $\Delta\phi(\omega)^2 = K\langle \phi(\omega)\rangle$ has been verified experimentally for several frequencies [30], where K was seen to be more or less independent of frequency. It is easy to show that

$$K = \left\langle \frac{d\phi}{d\omega} \right\rangle \int_{-\infty}^{\infty} d\Omega\, C_\phi(\Omega). \tag{15}$$

K is shown in Fig. 1 as a function of absorption. The value $K \approx 1.0$ near $L/L_a \approx 5$ coincides with the experimental value reported in Ref. [30]. Without absorption we find the somewhat larger value $K=1.61$.

IV. NON-GAUSSIAN FREQUENCY CORRELATIONS IN $I_D\phi/D\omega$

Consider the frequency correlation function of four field and introduce

$$\Delta(\nu,\omega,\Omega) \equiv \langle E_{\nu-\Omega/2-\omega/2} E^*_{\nu-\Omega/2+\omega/2} E_{\nu+\Omega/2+\omega/2} E^*_{\nu+\Omega/2-\omega/2} \rangle - \langle E_{\nu-\Omega/2-\omega/2} E^*_{\nu-\Omega/2+\omega/2} E_{\nu+\Omega/2-\omega/2} E^*_{\nu+\Omega/2+\omega/2} \rangle. \tag{16}$$

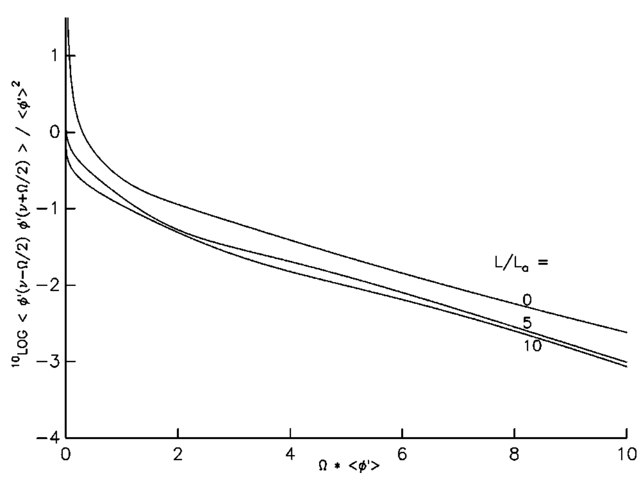

FIG. 2. Correlation function with frequency of the phase delay time $d\phi/d\omega$, on the basis of Gaussian theory. Different values for the absorption L/L_a have been considered. The frequency has been scaled with the ensemble average delay time.

The following identity is straightforward to prove:

$$\langle W_{\nu-\Omega/2}W_{\nu+\Omega/2}\rangle=\lim_{\omega\to 0}\frac{\Delta(\nu,\omega,\Omega)}{2\omega^2}. \tag{17}$$

In the Gaussian C_1 approximation, one decouples all field averages into the two-field average $C(\Omega)\equiv\langle E_{\nu-\Omega}E^*_{\nu+\Omega}\rangle$. Some algebra then confirms Eq. (11). We will use Eq. (17) to find the first non-Gaussian correction to this correlation function, to be referred to as $F_2(\Omega)$.

The standard recipe for calculating four-field correlators in the C_2 approximation has been described by Berkovits and Feng [4]. We will follow their analysis and notation, with the technical difference that we will deal with four (rather than two) different frequencies. The mechanism responsible for the non-Gaussian correlation is an exchange of momenta among the four fields, as described by the four-point Hikami box $H(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_5)$ (Fig. 3). Such an event makes different wave trajectories correlated and would be excluded by Gaussian statistics. The correct expression for the vertex H was given by Nieuwenhuizen and Van Rossum [33],

$$H(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_5)=-N_H(\boldsymbol{q}_1\cdot\boldsymbol{q}_2+\boldsymbol{q}_3\cdot\boldsymbol{q}_4). \tag{18}$$

Here $N_H=\ell^5/(48\pi k^5)$. If we denote the four-field correlation by $C_2(\Delta\nu_i)$ we have, in wave number space,

$$\begin{aligned}C_2(\{\boldsymbol{q}_i\},\{\Delta\nu_i\})=2\times C_1(\boldsymbol{q}_1,\Delta\nu_1)C_2(\boldsymbol{q}_2,\Delta\nu_2)\\ \times H(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_5)C_3(\boldsymbol{q}_3,\Delta\nu_3)\\ \times C_4(\boldsymbol{q}_4,\Delta\nu_4).\end{aligned} \tag{19}$$

For the object in Fig. 3 one has $\{\Delta\nu_i\}=\{\omega+\Omega,\omega-\Omega,\omega,\omega\}$. The Fourier transform of this object gives the correlations in real space, and finally correlations in transmission or reflection. Following Ref. [4] for transmission through a slab with length L and surface $A\gg L^2$, this procedure leads to

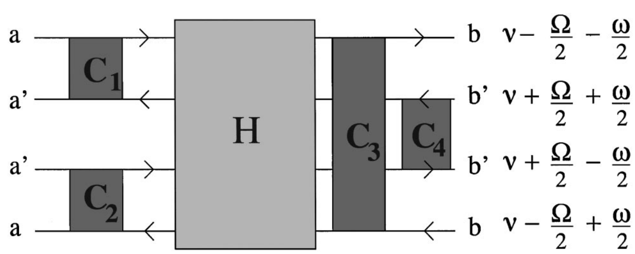

FIG. 3. Non-Gaussian "C_2" correlation of four complex fields [actually the second term in Eq. (16)], whose frequencies have been indicated on the right. The channels a and a' are two incident channels; b and b' are outgoing channels. In the present paper we assume $a=a'$ and $b=b'$. The Hikami box H formally locates the position where momentum exchange occurs between the four fields, giving non-Gaussian correlations. C_i denotes the two field correlator $\langle EE^*\rangle$, the labels indicating specific frequency difference ($\Omega+\omega$, $\omega-\Omega$, ω, ω, respectively). The mirror image of this diagram is not shown but contributes equally. The convention is that fields E propagate to the right and their complex conjugates E^* to the left.

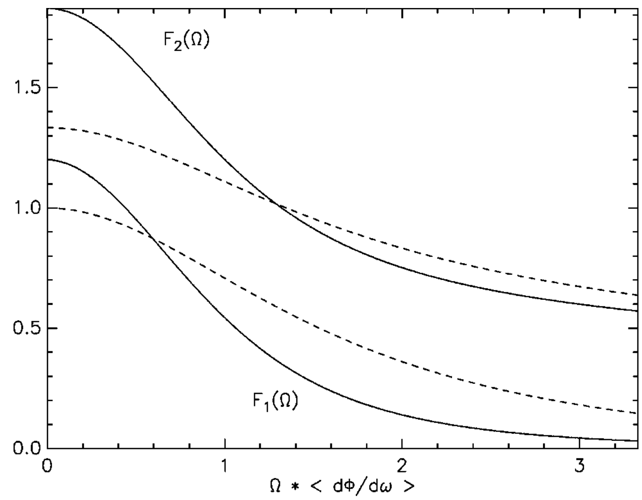

FIG. 4. Normalized correlations functions with frequency of the weighted phase delay time $Id\phi/d\omega$ in transmission (solid lines), evaluated for negligible absorption ($L/L_a=0$). The frequency difference Ω on the horizontal axis has been scaled with the ensemble averaged delay time. $F_1(\Omega)$ denotes the Gaussian C_1 approximation; $(1/g)\,F_2(\Omega)$ equals the non-Gaussian C_2 correction. For comparison, we have also shown the C_1 and C_2 frequency correlation $\langle I(\nu-\frac{1}{2}\Omega)I(\nu+\frac{1}{2}\Omega)\rangle$ for the intensity (dashed lines), as calculated in Ref. [4].

$$\begin{aligned}C_2^{\text{trans}}(\{\Delta\nu_i\})=2\left(\frac{2\pi}{k}\right)^4\left(\frac{\ell}{4\pi}\right)^4\frac{N_H}{A^4}\int d^3\boldsymbol{r}\\ \times\int d^2\boldsymbol{\rho}_1\cdots d^2\boldsymbol{\rho}_4 C_2(\{\boldsymbol{\rho}_i\},\{z_i\},\boldsymbol{r};\{\Delta\nu_i\}).\end{aligned} \tag{20}$$

In this equation $\boldsymbol{r}$ denotes the position of the Hikami vertex in the medium, and $\{z_i\}=\{\ell,\ell,L-\ell,L-\ell\}$ denote approximate entrance and exit depths of the four waves in the slab. Equation (20) can be transformed into

$$\begin{aligned}C_2^{\text{trans}}(\{\Delta\nu_i\})=|\langle T_{ab}\rangle|^2\frac{2}{g}[I(\beta_1,\beta_2,\beta_3,\beta_4)\\ +I(\beta_3,\beta_3,\beta_1,\beta_2)],\end{aligned} \tag{21}$$

with $\beta_i\equiv\sqrt{-i\Delta\nu_i L^2/D+(L/L_a)^2}$ and

$$\begin{aligned}I(\beta_1,\beta_2,\beta_3,\beta_4)\equiv\frac{\beta_1\beta_2}{\sinh\beta_1\cdots\sinh\beta_4}\\ \times\int_0^1 dx\,\cosh\beta_1(1-x)\cosh\beta_2(1-x)\\ \times\sinh\beta_3 x\,\sinh\beta_4 x.\end{aligned} \tag{22}$$

We have also introduced the average one-channel transmission coefficient $\langle T_{ab}\rangle=(3\pi/Ak^2)\ell/L$ and the dimensionless conductance $g=(Ak^2/3\pi)\ell/L$. The normalized C_2 frequency correlation function $F_2(\Omega)$ for W_{ab} in transmission can now be obtained from Eq. (17),

$$F_2(\Omega) = \lim_{\omega \to 0} \frac{C_2^{\text{trans}}(\Omega,-\Omega,\omega,-\omega) - C_2^{\text{trans}}(\Omega+\omega,\omega-\Omega,\omega,\omega)}{\omega^2}. \tag{23}$$

The limit can be carried out analytically. What remains is an integral of the kind (22) that is easily done numerically.

The special case $\Omega = 0$ with no absorption ($L_a = \infty$) can be handled analytically. The result is

$$\langle \hat{W}_{ab}^2 \rangle = \frac{11}{5} + \frac{64}{35g} + O\left(\frac{1}{g^2}\right). \tag{24}$$

This can be compared to the similar expression for $\langle \hat{I}_{ab}^2 \rangle = 2 + 4/3g$. It can be inferred that delay-time fluctuations cause the fluctuations in W to exceed those ones in I, in both the C_1 and C_2 approximations. In Fig. 1 we show $F_2(0)$ as a function of absorption. We notice that F_2 is more sensitive to absorption than F_1.

In Fig. 4 we show several frequency correlation functions for $\hat{W}_{ab}$ (solid lines) in transmission of a slab without absorption. In contrast to the correlation function of ϕ', no singularity is encountered at $\Omega = 0$. Also shown (dashed) are the correlation functions of the intensity I_{ab}. These correlation functions are similar, but the correlation of $I_{ab}\phi'_{ab}$ exceeds that of I_{ab} at small frequency differences, whereas it becomes smaller at large frequencies differences. The F_2 correlation function decays as $1/\sqrt{\Omega\langle \phi' \rangle}$, quite similar to the long-range correlation function of the intensity [5]. In Ref. [1] this non-Gaussian theory was shown to agree with microwave measurements.

V. CONCLUSIONS AND PROSPECTS

In conclusion, we have presented a theory for the statistics of the phase delay time of diffusing waves starting from a joint Gaussian distribution for complex fields. The sensitivity of phase to parameters other than frequency (e.g., space, time, or magnetic field) may be a useful application of this theory. We have also presented a non-Gaussian extension for the statistics of $I d\phi/d\omega$, i.e., the intensity weighted delay time. The overall conclusion is that the delay time exhibits large mesoscopic fluctuations. Multichannel correlations, the statistics of $N(\omega)$, and a comparison to random matrix theory are logical continuations of this work. We consider the frequency derivative of the phase, which equals the single channel delay time in the narrow band limit, to be the basic statistical dynamical variable. It is a great challenge to study the statistics of arbitrary wave packets.

ACKNOWLEDGMENTS

We thank Frédéric Faure, Roger Maynard, Ad Lagendijk, and Gabriel Cwilich for discussions. This work was supported by the Groupement de Recherches POAN and the National Science Foundation under Grant Nos. DMR 9632789 and INT9512975.

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