

## Field distributions in the crossover from ballistic to diffusive wave propagation

A. A. Chabanov and A. Z. Genack

*Department of Physics, Queens College, CUNY, and New York State Center for Advanced Technology for Ultrafast Photonic Materials and Applications, Flushing, New York 11367*

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We have measured the probability distributions of a microwave field transmitted through a random medium in the crossover from ballistic to diffusive propagation. The joint distribution of in-phase and out-of-phase components of the field is nearly Gaussian, even in samples with thickness less than the scattering mean free path. In these samples, the phase distribution of the residual field, which is the difference between the field and its ensemble average, is nearly flat. The extent of the deviation of the residual field from complete randomization depends upon the frequency, sample length, and modes of excitation. [S1063-651X(97)51608-4]

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A wave propagating in a random medium loses spatial coherence as a result of interference between scattered waves. The randomization of the incident coherent field is of considerable current interest because of the increasing use of classical waves to image multiply scattering samples. We study the randomization of the incident field by measuring statistical characteristics of continuous waves transmitted through an ensemble of random scatterer configurations in the crossover from ballistic to diffusive propagation. The cw field  $\mathcal{E}(\mathbf{r}, t)$  at the receiver is a real function of position and time, and can be expressed in terms of the complex amplitude  $\mathbf{E}(\mathbf{r})$ ,

$$\mathcal{E}(\mathbf{r}, t) = \text{Re}[\mathbf{E}(\mathbf{r})\exp(i\omega_0 t)], \quad \mathbf{E}(\mathbf{r}) = E(\mathbf{r})\exp[i\varphi(\mathbf{r})], \quad (1)$$

where  $\omega_0$  is a carrier frequency. The complex field  $\mathbf{E}(\mathbf{r})$ , as a random function of position, can be written as the sum of the average and residual fields,

$$\mathbf{E}(\mathbf{r}) = \langle \mathbf{E}(\mathbf{r}) \rangle + \delta\mathbf{E}(\mathbf{r}), \quad (2)$$

where the angle brackets represent the ensemble average. In this article, we present the statistical properties of both  $\mathbf{E}(\mathbf{r})$  and  $\delta\mathbf{E}(\mathbf{r})$ .

The statistics of the intensity in the crossover from ballistic to diffusive transport have been calculated recently by Kogan and Kaveh using diagrammatic techniques [1]. They describe intensity fluctuations in terms of two kinds of diagrams, ‘‘ladder’’ diagrams and diagrams containing ‘‘dressed’’ propagators, and obtain the intensity distribution from a moment expansion of its characteristic function. Though the distribution is given in the form of an integral in [1], it can be evaluated [2] and yields

$$P(u) = \frac{1}{1-a} \exp\left(\frac{u+a}{1-a}\right) \mathcal{I}_0\left(2\sqrt{\frac{ua}{1-a}}\right). \quad (3)$$

In Eq. (3),  $u = I/\langle I \rangle$  and  $a = I_c/\langle I \rangle$ , where  $\langle I \rangle$  and  $I_c = |\langle \mathbf{E}(\mathbf{r}) \rangle|^2$  are the average and coherent intensities, respectively, and  $\mathcal{I}_0$  is a modified Bessel function of the first kind of zero order.

To obtain the field distributions, we need to find an appropriate model for  $\mathbf{E}(\mathbf{r})$  that treats the phase as well as the

amplitude. The ‘‘ladder’’ diagrams ignore interference between partial fields associated with Feynman paths which reach the point  $\mathbf{r}$ . More precisely, they only account for statistically independent paths, such that any phase difference between them is suppressed by the averaging. Thus the ‘‘ladder’’ diagram approximation is equivalent to the model of a random phasor sum which is frequently used in statistical optics [3,4]. On the other hand, the diagrams containing ‘‘dressed’’ propagators represent the decaying coherent intensity  $I_c$  [5]. Hence, the model of a constant phasor plus a random phasor sum may serve to extend the results of the diagrammatic approach of [1] to obtain the statistics of  $\mathbf{E}(\mathbf{r})$ . Indeed, the intensity distribution (3) is precisely the result of this model when the random phasor sum is a *circular* random variable (see, for example, [3]).

The model of a random phasor sum is based upon the assumption that a large number of statistically independent phasors contribute to the resultant field. However, one might expect in the case of an optically thin sample excited by a single mode (e.g., plane wave), that the fluctuations of the field in the incident mode are much larger than those of the other modes. As a result, there would be departures from Gaussian statistics of the model. On the other hand, when the incident intensity is evenly distributed over a number of incident modes (e.g., in a waveguide geometry), none of the fluctuations of these modes dominates and the departures from the Gaussian statistics would be weaker.

The sample is a collection of randomly positioned,  $\frac{1}{2}$ -in.-diam polystyrene spheres at a volume fraction of 0.55. The sample is contained within a 7.5-cm-diam copper tube, and its length is adjusted using a plunger. Field measurements are performed as the tube is rotated about its axis at 1.6 Hz, so that the spheres change their positions and new sample configurations are produced. Linearly polarized cw microwave radiation is launched from a horn placed 0.3 m in front of the sample along the tube axis. A wire probe detector oriented parallel to the incident polarization is attached to the plunger inside the tube at its center. Measurements of the field are made using a Hewlett-Packard 8722C network vector analyzer. The analyzer samples 1600 data points at uniform time intervals during each 30 s sweep. The sampling time is about  $\frac{1}{3}$  ms, which is much smaller than the correlation time of the residual field for the optically longest

sample,  $\tau_c \approx 3$  ms. So, data are taken on a time scale in which the sample may be considered as static. Approximately  $10^6$  statistically independent field measurements are taken at each sample length. Measurements are made at 11, 15, and 19 GHz at sample lengths from 0.1 to 0.4 m.

We measure the real and imaginary parts  $r = \text{Re}[\mathbf{E}(\mathbf{r})]$  and  $i = \text{Im}[\mathbf{E}(\mathbf{r})]$  of the complex field  $\mathbf{E}(\mathbf{r})$  at a point in the near-field speckle pattern of the transmitted wave for an ensemble of random scatterer configurations. From these measurements, we obtain the joint probability distribution function of the real and imaginary parts of the transmitted field,  $P(r, i)$ . We also study the statistical properties of the residual field  $\delta\mathbf{E}(\mathbf{r})$ . The experimental results are compared with predictions of the model of a constant phasor plus a random phasor sum and the diagrammatic approach of [1].

In the random-phasor-sum model the complex field  $\mathbf{E}$  is assumed to be a superposition of a large number of statistically independent phasors, each having random amplitude  $e_k$  and random phase  $\phi_k$ , [4]

$$\mathbf{E} = E \exp(i\varphi) = \sum_k e_k \exp(i\phi_k). \quad (4)$$

By virtue of the central limit theorem, the real and imaginary parts of the resultant field obey Gaussian statistics, so that their joint probability density function  $P_g(r, i)$  can be obtained from the first- and second-order joint moments of  $r$  and  $i$ ,

$$P_g(r, i) = \frac{1}{2\pi\sigma_0^2} \exp \left[ -\frac{1}{2\sigma_0^4} (\sigma_{ii}^2 (r - \langle r \rangle)^2 + \sigma_{rr}^2 (i - \langle i \rangle)^2 - \sigma_{ri}^2 (r - \langle r \rangle)(i - \langle i \rangle)) \right]. \quad (5)$$

Here  $\sigma_0^4 = \sigma_{rr}^2 \sigma_{ii}^2 - \sigma_{ri}^4$ , and  $\sigma_{rr}^2 = \langle (r - \langle r \rangle)^2 \rangle$ ,  $\sigma_{ii}^2 = \langle (i - \langle i \rangle)^2 \rangle$ , and  $\sigma_{ri}^2 = \langle (r - \langle r \rangle)(i - \langle i \rangle) \rangle$ , respectively. If the phases  $\phi_k$  are uniformly distributed on the primary interval  $(-\pi, \pi)$ , the real and imaginary parts of the resultant field are statistically independent and identically distributed with zero mean, i.e.,  $\langle r \rangle = \langle i \rangle = 0$ ,  $\sigma_{rr}^2 = \sigma_{ii}^2 = \sigma^2$ ,  $\sigma_{ri}^2 = 0$ , and

$$P_g(r, i) = \frac{1}{2\pi\sigma^2} \exp \left( -\frac{r^2 + i^2}{2\sigma^2} \right). \quad (6)$$

Contours of constant probability density are then circles in the  $(r, i)$  plane, and the resultant field  $\mathbf{E}$  is a *circular* random variable. In general, if the distributions of  $\phi_k$  are other than uniform,  $r$  and  $i$  do not have zero mean and the contours of constant probability density are predicted to be ellipses in the complex plane. Further, if a constant phasor is added to the random phasor sum (4), the distributions (5) and (6) are shifted as a whole in the  $(r, i)$  plane by the vector corresponding to the constant phasor.

Contours of constant probability density of the distributions  $P(r, i)$  measured in random samples of length  $L = 0.3$  m at 19 GHz and  $L = 0.1$  m at 11 GHz are shown in Figs.

1(a) and 1(b), respectively, as the thick curves. In Fig. 1(a),  $a = 0.01$ . The contours are nearly circles in the  $(r, i)$  plane, centered at the point corresponding to the average field in agreement with the model of a constant phasor plus a (circular) random phasor sum. The distribution functions of the normalized amplitude  $A = E/\sigma$  and phase of the field transmitted through the longer sample are presented by squares in Figs. 2(a) and 2(b), respectively. The solid lines plotted through the data are the distributions derived in the framework of the model [4],

$$P(A, k) = A \exp \left( -\frac{A^2 + k^2}{2} \right) \mathcal{I}_0(Ak), \quad (7)$$

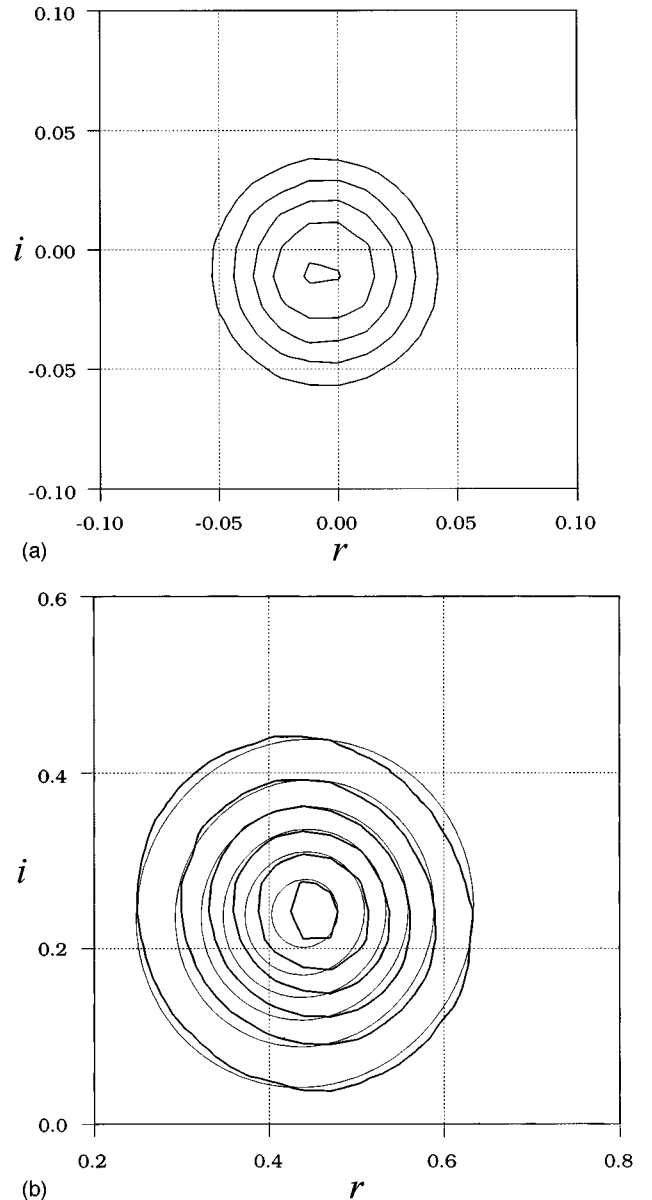


FIG. 1. Contours of constant probability density  $P(r, i) = \text{const.}$ : (a)  $L = 0.3$  m,  $f = 19$  GHz ( $a = 0.01$ ); (b)  $L = 0.1$  m,  $f = 11$  GHz ( $a = 0.94$ ). Thin curves in (b) represent the Gaussian distribution  $P_g(r, i)$  of Eq. (5) calculated using the measured first- and second-order joint moments of  $r$  and  $i$ .

$$P(\varphi, k) = \frac{1}{2\pi} \exp\left(-\frac{k^2}{2}\right) + \frac{k \cos\varphi}{\sqrt{2\pi}} \times \exp\left(-\frac{(k \sin\varphi)^2}{2}\right) \text{erf}(k \cos\varphi), \quad (8)$$

where  $\text{erf}(x)$  is an error function. The distributions (7) and (8) are functions of the measured parameter  $k = |\langle \mathbf{E}(\mathbf{r}) \rangle| / \sigma$ , which is related to  $a$  of Eqs. (3) by  $k^2 = 2a/(1-a)$ . Using the measured value  $k = 0.15$ , we obtain excellent agreement between the experimental and theoretical distributions. The amplitude and phase distributions for the residual field  $\delta \mathbf{E}(\mathbf{r})$

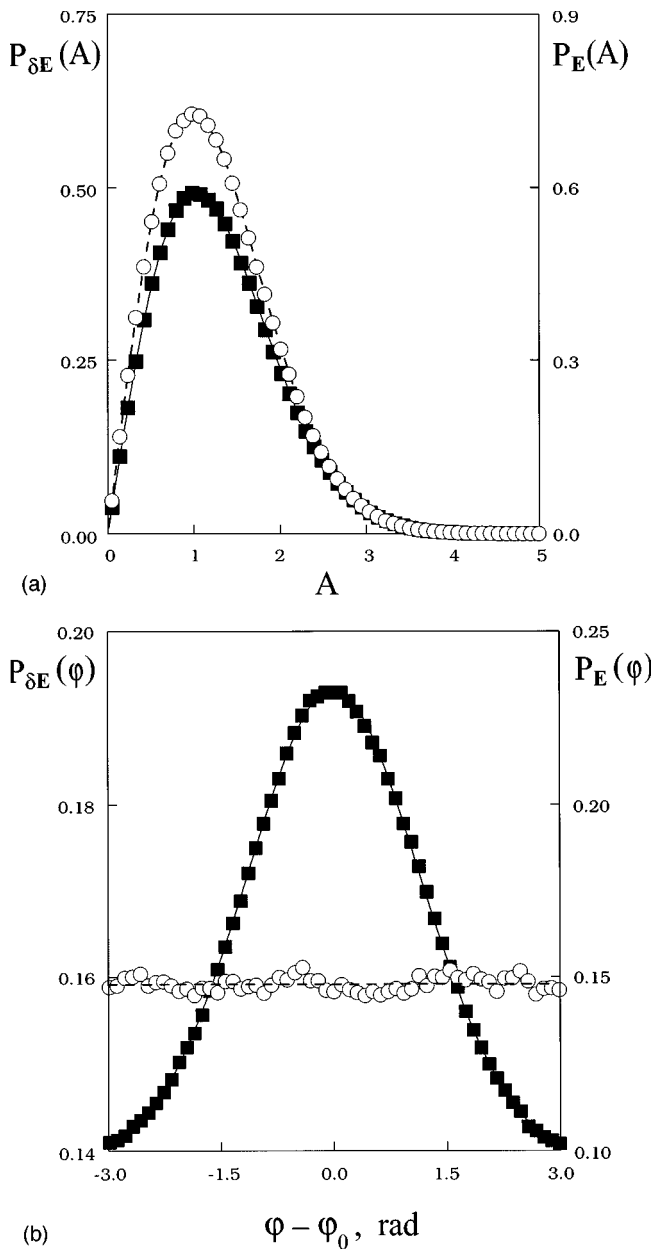


FIG. 2. Distributions of the normalized amplitude (a) and phase (b) of the total field (squares) and the residual field (circle) in a sample of  $L=0.3$  m at 19 GHz. Solid and dashed lines are the theoretical distributions for  $k=0.15$  and 0, respectively. The phase distributions are plotted about the phase of the average field,  $\varphi_0 = \arg\langle \mathbf{E} \rangle$ .

are also shown as the circles in Figs. 2(a) and 2(b). The corresponding theoretical curves (dashed lines) are obtained from Eqs. (7) and (8) with  $k=0$ . In this case, Eq. (7) gives the Rayleigh distribution for the amplitude and Eq. (8) a uniform distribution for the phase.

For shorter samples or lower frequencies, as the parameter  $a$  approaches unity, deviations from a circular distribution of  $P(r, i)$  appear. In Fig. 1(b),  $a=0.94$ . The contours of constant probability density appear to be elliptical. However, the measured distribution  $P(r, i)$  also differs appreciably from the Gaussian distribution  $P_g(r, i)$  of Eq. (5), calculated from the first- and second-order joint moments of  $r$  and  $i$ , and shown in Fig. 1(b) as the thin curves. The amplitude and

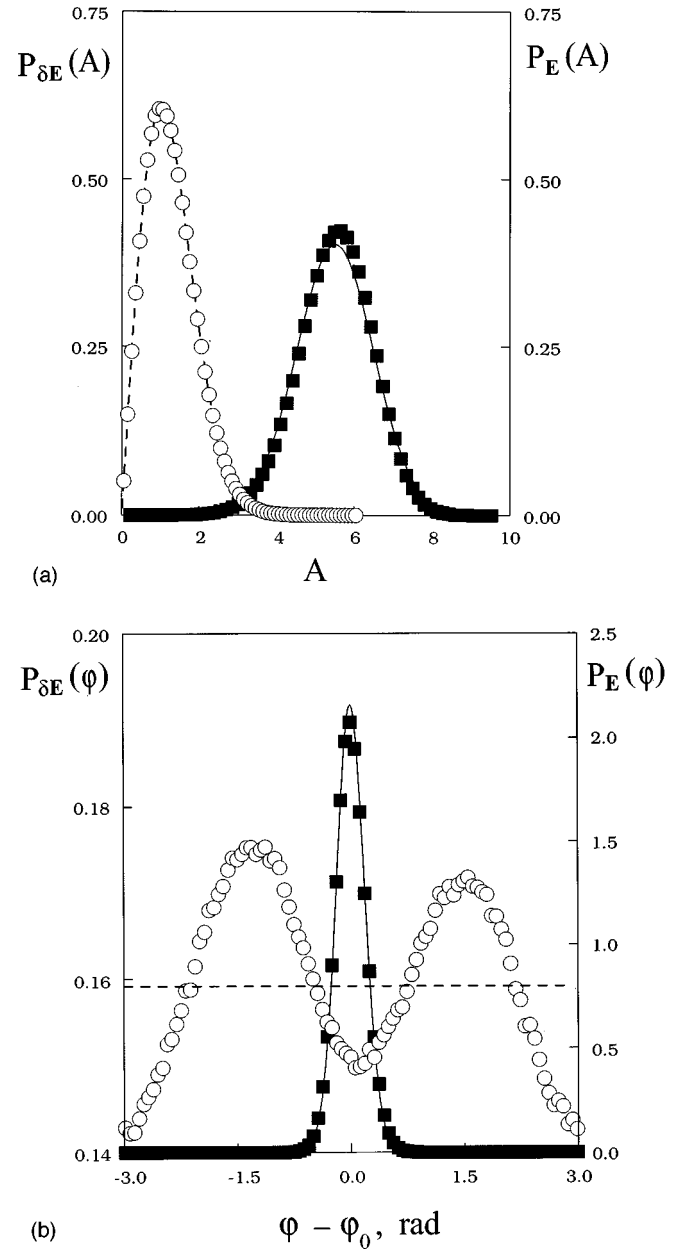


FIG. 3. Distributions of the normalized amplitude (a) and phase (b) of the total field (squares) and the residual field (circle) in a sample of  $L=0.1$  m at 11 GHz. Solid and dashed lines are the theoretical distributions for  $k=5.4$  and 0, respectively. The phase distributions are plotted about the phase of the average field,  $\varphi_0 = \arg\langle \mathbf{E} \rangle$ .

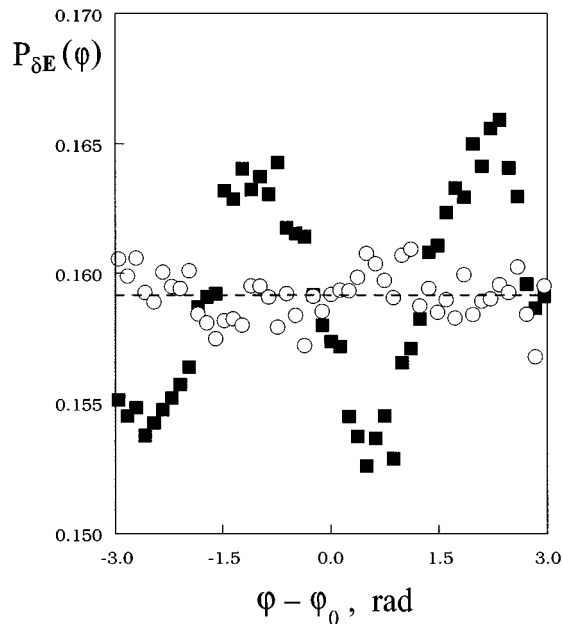


FIG. 4. Phase distribution of the residual field for a sample of  $L=0.2$  m at 15 GHz with a horn (squares) and wire antenna (circles) sources. The parameter  $a=0.66$  and  $0.16$ , respectively.

phase statistics of the total and residual fields are presented in Figs. 3(a) and 3(b), respectively. Deviations are seen from the corresponding theoretical curves calculated assuming the residual field is randomized. The normalized amplitude distribution of the residual field  $P_{\delta E}(A)$  in Fig. 3(a), however, is still close to the Rayleigh distribution function.

The model of a constant phasor plus a (circular) random phasor sum and the diagrammatic approach of [1] accurately describes our measurements of the statistics of the transmitted field in the crossover from ballistic to diffusive wave propagation. The departures from the circular distribution and from the Gaussian distribution (5) resulting from violation of the assumption that a large number of independent random phasors contribute to  $\mathbf{E}(\mathbf{r})$  are only observed in samples where the parameter  $a$  is near unity. However, we believe that the good agreement between the measured sta-

tistics and the results of the model is a consequence of a large number of waveguide modes excited in our sample, and that the deviations would be larger, if only a single mode were excited by the incident wave.

The wave generated by the horn excites a number of low order waveguide modes. When the horn is replaced by a wire antenna positioned just in front of the sample, the incident wave couples to a larger number of modes. Because we expect a significant dependence of the field statistics upon the number of the incident modes, we have compared the statistics of the fields obtained with a horn and with a wire antenna as sources in a sample of length  $L=0.2$  m at 15 GHz. The phase statistics of the residual field shown in Fig. 4 are in better agreement with the model [Eq. (7) with  $k=0$ ] in the case of a wire source. However, we should note that the parameter  $a$  in the case of the wire source is appreciably reduced, and hence the comparison is suggestive only. The reduction in the parameter  $a$  may result from more effective scattering, because of the greater dwell time for higher order waveguide modes, and because of interference between these modes.

In conclusion, the statistics of the field transmitted through optically thin random media are found to be in remarkably good agreement with predictions of the model of a constant phasor plus a random phasor sum and diagrammatic approach of [1]. This may be a consequence of the excitation of a number of modes by the incident field. We believe that larger deviations could be observed in geometries in which the number of coherent modes excited can be reduced. This is readily achieved in optical experiments or in measurements in any system in which the transverse dimensions of the sample are much larger than the wavelength. The simplicity of the distributions observed may prove to be useful for separating ballistic from diffusive radiation in imaging applications.

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