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Formation of positive feedback and coherent emission in a cavity-free system

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Abstract: We develop a theory of lasing of a collection of pumped active atoms without a resonator (either regular or random). Due to spontaneous emission into free space, phases of free space electromagnetic modes fluctuate. These phase fluctuations can be reduced to frequency fluctuations. The closer the frequency of fluctuation to the transition frequency of the active atoms, the higher lifetime of the fluctuation. We show that because of this, the average frequency of modes pulls toward the transition frequency. This leads to a maximum in the density of states of the electromagnetic field and a decrease of the mode group velocity. Consequently, the coupling of modes with atoms as well as the lifetime of fluctuations increase. Thus, mode pulling provides positive feedback. When the pump rate exceeds a certain threshold, the lifetime of one of the realized fluctuations diverges, and radiation becomes coherent.

1. Introduction

Radiation of a collection of incoherently pumped atoms in a cavity-free system is usually referred to as amplified spontaneous emission (ASE) or superfluorescence [1]. It occurs in various physical systems – from so-called cosmic lasers to superluminescent diodes [2–8]. In an ASE system, a photon spontaneously emitted by one of the excited atoms triggers stimulated emission of inverted atoms located on its path out of the system [1]. Subsequently, a pulse of coherent photons is formed. Since for the ASE system it is assumed that there is no reflection at the system boundaries, this pulse leaves the system. Total radiation of an ASE system is a sum of such pulses that have random phases. For a small gain coefficient, the resulted radiation is incoherent [9].

A laser is an ASE system placed in a cavity (resonator). In a laser, an ASE pulse, which is formed inside the cavity, reaches the resonator boundary, returns back, and is amplified further. The resonator provides positive feedback that makes a laser generate coherent radiation. Indeed, in a laser, the pulse with the maximum number of photons has an advantage in further amplification; when the phase and energy conditions are satisfied [1], the system lases.

A resonator does not have to be a part of a laser structure. In random lasers, photons scattered on inhomogeneities return to the gain medium. As a result, the path of the photon through the system becomes complicated. Multiple scatterings can lead to backscattering, similar to weak localization. Configurations of the disordered medium causing backscattering can be considered as random mirrors [10]. Moreover, a closed path may be formed. This means that in the system, several random mirrors form a random resonator. It is understood that when random resonators are present, they provide positive feedback, and can lead to lasing when phase conditions are satisfied [11–13]. With no closed paths, a random laser should behave as an ASE system [12].

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Below we deal with systems that have no backscattering; then, even a random system is equivalent to an ASE system. As far as we know, it is commonly expected that an ASE system can only serve as an incoherent light source or as a light amplifier (see [9] for detail). ASE and lasing, however, have some common features. First, in both lasing and ASE systems, line narrowing of output radiation is observed. Second, similar to lasing, the ASE intensity output has an S-shaped dependence on the pump rate. These features arise due to the nonlinear dependence of the radiation amplification on the pump rate.

In this paper, we show that even without a cavity, there is a mechanism for creating positive feedback in ASE systems. Because of spontaneous emission of atoms into modes, the complex amplitudes of the modes fluctuate. The autocorrelation function of these fluctuations shows the spectrum broadening that can be treated as frequency fluctuations. Due to the interaction of free-space modes with pumped atoms, the closer the mode frequency to the transition frequency of pumped atoms, the longer the lifetime of the fluctuation. This results in pulling mean frequencies of free-space modes toward the transition frequency. In turn, this leads to an increase in the density of states (DOS) near the transition frequency. As a result, the group velocity decreases, and the interaction between the modes and atoms increases. Consequently, the intensity of the modes grows to cause further enhancement of the frequency pulling. By computer simulation, we find the parameters of a cavity-free system for which the lifetime of such fluctuations may diverge so that the stimulated emission of the inverted atoms results in self-oscillations (lasing). The latter is verified by the calculation of the second-order correlation function, which tends to unity with pumping increase.

2. The model

In a typical ASE experiment, a large volume is usually filled with active atoms. Only a fraction of these atoms within a region stretched in one direction is pumped. It is the direction, in which ASE is observed. Thus, for simplicity, we consider a one-dimensional multimode waveguide extended in the x-direction. Only a part of this waveguide within the length L_{am} is filled with pumped atoms. To make this model closer to a real ASE experiment, we take into account radiation losses in the $1D$ waveguide walls that correspond to losses through side boundaries of the $3D$ pumped region.

In order to quantize the electromagnetic (EM) field, we assume that the waveguide is bounded by ideally reflecting walls, separated by a very large distance L_U acting as the size of the Universe. We assume that the empty waveguide has many eigenmodes, which are standing waves. Note that the Poynting vector of any waveguide mode is equal to zero, and radiation into this mode would violate causality. Since we take into account a large number of modes, the interference in these modes forms a sharp front propagating with the speed of light [14,15] and results in a non-zero Poynting vector. To avoid the effect of the return of the front reflected from the boundary of the Universe into the active volume, we consider time-scales smaller than the round-trip time of light $t_U = L_U/c$. By increasing L_U, we make t_U greater than any time of a transient process in the system.

In our computer simulation, we use the Maxwell-Bloch equations with quantum noise (for details, see [9,16,17]):

$$\frac{d}{dt}a_j = (-\gamma_a/2 - i\Delta_j)a_j - i\sum_{k}^{atoms}\Omega_{jk}\sigma_k, \tag{1}$$

$$\frac{d}{dt}\sigma_k = -\sigma_k(\gamma_P + \gamma_D + \gamma_{deph})/2 + iD_k\sum_{j}^{modes}\Omega_{jk}a_j + F_k^{\sigma}, \tag{2}$$

$$\frac{d}{dt}D_k = (\gamma_P - \gamma_D) - (\gamma_P + \gamma_D)D_k + 2i\sum_{j}^{modes}\Omega_{jk}(a_j^*\sigma_k - a_j\sigma_k^*), \tag{3}$$

where a_j is the amplitude of the EM field in the j-th free-space mode, σ_k and D_k are the polarization and the population inversion of the k-th atom, respectively. Also γ_a is the relaxation rate that describes energy losses in the waveguide walls, γ_D and γ_{deph} are energy and phase relaxations rates of atoms, respectively, and γ_P is the rate of incoherent pumping of the atoms; $\Delta_j = \omega_j - \omega_{TLS}$ is the difference between the eigenfrequency of the j-th mode and the transition frequency of the atoms. Since the eigenmodes are standing, the coupling constant (the Rabi frequency) between the j-th mode and the k-th atom, Ω_{jk}, is real. The value of Ω_{jk} is equal to $-\mathbf{E}_j(x_k) \cdot \mathbf{d}_k/\hbar$, where x_k is the position of the k-th atoms, $\mathbf{d}_k = \langle e|e\mathbf{r}|g\rangle_k$ is the matrix element of its dipole moment, and $\mathbf{E}_j(x)$ is the electric field "per one photon" of the j-th mode (for details, see [9,16,18]). F_k^σ are the noise terms. As has been shown in [16,19], the noise terms added to Eqs. (1)–(3) allow one to describe phenomena associated with spontaneous emission. Note that in Eq. (3), the last sum contains terms that are responsible for the absorption of photons from the modes. These terms are needed because, in a system of many atoms that we consider, they describe processes in which a photon emitted by one atom is absorbed by another one.

The values of the parameters that we consider are close to those of a gain medium based on organic semiconductors [20]. The atoms are modeled by two-level systems (TLSs) with the transition frequency $\omega_{TLS} \sim 3 \cdot 10^{15} s^{-1}$, $\lambda_{TLS} = 2\pi c/\omega_{TLS} \sim 670$ nm. The relaxation rates that describe energy losses in the waveguide walls and the active medium are $\gamma_a = 4 \times 10^{-3}\, \omega_{TLS}$ and $\gamma_D = 10^{-6}\, \omega_{TLS}$, respectively. The rate of the phase relaxation of the atoms is $\gamma_{deph} = 10^{-2}\, \omega_{TLS}$; the rate of incoherent pumping γ_P varies within the interval $(0,\ 1000\gamma_D)$.

The gain of pumped atoms is characterized by the gain coefficient $G(n) = 4\pi n \omega_{TLS}|d|^2/\hbar c \gamma_{deph} = (2.3 \cdot 10^{-15}\, cm^2)\, n$, where d is the dipole moment of a TLS transition and $n \sim 6.4 \div 32 \cdot 10^{17}\ cm^{-3}$ is the density of active atoms, $L_{am} = 35\lambda_{TLS}$. Since the interaction of modes with atoms is resonant, we consider a finite frequency interval $(\omega_{TLS} - 20\gamma_\sigma,\ \omega_{TLS} + 20\gamma_\sigma)$. The value of L_U is chosen large enough to provide about 1600 equidistant modes within the interval. The noise terms and the relaxation rates in Eqs. (1)–(3) are connected via the fluctuation-dissipation theorem [18]. Since the phase relaxation rate of dipole moments, γ_{deph}, is much greater than the other relaxation rates, corresponding noise terms, F_k^σ, prevail [18]. For this reason, we take into account only these noise terms.

3. Results of computer simulation

Our computer simulation shows that the parameter determining the dynamics of the system is the product $G(n)L_{am}$. For $G(n)L_{am} = 5.2$ ($n = 6.4 \times 10^{17}\ cm^{-3}$), we obtain that the dependence of the output radiation on the pump rate has a characteristic S-shape [1]. The pump rate that corresponds to the inflection point of the curve is usually considered as the ASE-threshold, γ_{ASE} [7,8]. An increase in gain does not change the shape of the curve qualitatively. We find that the only characteristic of the system, which does change qualitatively, is the coherence of output radiation.

Coherent properties of light are characterized by the second-order coherence function:

$$g^{(2)}(\tau) = \langle I(t)I(t+\tau)\rangle / \langle I(t)\rangle^2, \tag{4}$$

where I is the radiation intensity. For incoherent light of a black body, $g^{(2)}(0) = 2$, while for lasers, $g^{(2)}(0) = 1$ [19]. For an ASE system, $g^{(2)}(\omega_{TLS}, 0)$ does not depend on the pump rate γ_P and is about 2 both below and above the threshold γ_{ASE} [9,21,22]. When $\tau \to \infty$, $g^{(2)}(\tau) \to 1.0$ with the characteristic time inversely proportional to the linewidth [16,19]. In experiments, to measure $g^{(2)}(0)$, a spectrum of the investigated source is narrowed by filtering [19], we, therefore, calculate $g^{(2)}(\omega_{TLS}, 0)$ in a frequency interval near the transition frequency of active atoms.

For relatively small $G(n)L_{am}$<8, the value of $g^{(2)}(\omega_{TLS}, 0)$ does not depend on the pump rate. In particular, for $G(n)L_{am} = 5.2$, $g^{(2)}(\omega_{TLS}, 0)$ is equal to 1.9 ± 0.1 (see also [9]) that characterizes the output radiation as incoherent.

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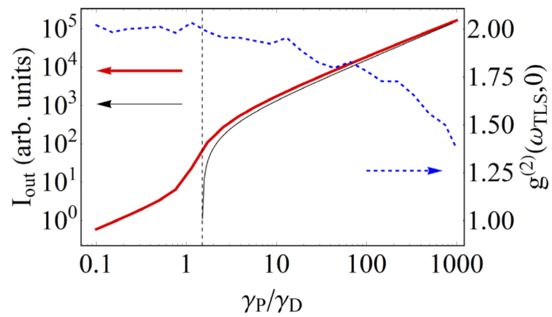

Fig. 1. The dependence of the intensity of output radiation and $g^{(2)}(\omega_{TLS}, 0)$ on the pump rate for the cavity-free system exhibiting the coherence threshold γ_{coh} (shown by the vertical dashed line). The solid thick and the thin curves are output intensities obtained by solving the Maxwell-Bloch equation with and without noise, respectively. The computer simulation is performed for $G(N_c)L_{am} = 26$ ($n = 32.0 \times 10^{17}\ cm^{-3}$). The dashed blue curve is the second-order correlation function, $g^{(2)}(\omega_{TLS}, 0)$.

For larger values of $G(n)L_{am}$, our model demonstrates an unexpected behavior of a cavity-free ASE system. We find that on the intensity curve, a new pumping threshold $\gamma_{coh}>\gamma_{ASE}$ arises (see Fig. 1). Above this threshold, the value of $g^{(2)}(\omega_{TLS}, 0)$ drops to unity (Fig. 1), which is typical for coherent light of lasers. This is a consequence of adding coherent radiation to the incoherent radiation arising in the system with noise, the appearance of which above the threshold is predicted in the problem without noise.

The system under consideration demonstrates the behavior that is typical for a common laser with cavity [1,16,19]. Since the refractive index of the active medium differs from the refractive index of the surrounding medium, one can expect that the lasing may occur at the Fabry-Perot cavity formed by the boundaries of the active medium. However, the estimation for the lasing threshold shows that lasing at the Fabry-Perot cavity requires the pump rate greater than γ_{coh} by several orders of magnitude. Indeed, the edges of the active medium play the role of mirrors, and lasing is determined by the reflection coefficient, which depends on dielectric permittivity of the active medium, $\varepsilon_{gain}(\omega)$ [23]:

$$\frac{\sqrt{\varepsilon_{gain}(\omega)} - 1}{\sqrt{\varepsilon_{gain}(\omega)} + 1} \exp\left(i\frac{\omega}{c}\sqrt{\varepsilon_{gain}(\omega)}\, L_{am}\right) = 1 \tag{5}$$

(the dielectric permittivity of the waveguide materials is assumed to be 1). The dielectric permittivity of the active medium below the lasing threshold can be estimated through the dipole moment of a TLS transition, d, and the concentration of TLSs, n, [23]:

$$\varepsilon_{gain}(\omega) = 1 - \frac{\alpha}{\omega_{TLS} - \omega - i\gamma_{deph}/2}, \tag{6}$$

where $\alpha = 4\pi|d|^2 n/\hbar$. The gain coefficient of the active medium is expressed via dielectric permittivity as [24]:

$$G = -2\frac{\omega}{c}\mathrm{Im}\sqrt{\varepsilon_{gain}(\omega)} \approx -\frac{\omega}{c}\mathrm{Im}\,\varepsilon_{gain}(\omega). \tag{7}$$

The absolute value of the reflection coefficient at the transition frequency, ω_{TLS}, is

$$|r(\omega_{TLS})| = \left|\frac{\sqrt{\varepsilon_{gain}(\omega_{TLS})} - 1}{\sqrt{\varepsilon_{gain}(\omega_{TLS})} + 1}\right| \approx \frac{|\mathrm{Im}\varepsilon_{gain}(\omega_{TLS})|}{4} \approx \frac{c\,G}{4\,\omega_{TLS}}. \tag{8}$$

At the coherent threshold, $G_{coh} = G D_{0\_coh} = 1450\, cm^{-1}$, from Eq. (8), we obtain that $|r(\omega_{TLS})| = 3.68 \cdot 10^{-3}$, where $D_{0\_coh} = (\gamma_{coh} - \gamma_D)/(\gamma_{coh} + \gamma_D)$. This value is much smaller than the absolute value of the reflection coefficient, which is necessary for lasing at the Fabry-Perot cavity

$$|r_{FP}(\omega_{TLS})| = \left|\exp\left(i\frac{\omega}{c}\sqrt{\varepsilon_{gain}(\omega)}\, L_{am}\right)\right|^{-1} = \exp(-G_{coh}\, L_{am}/2) \approx 7.9 \cdot 10^{-2}. \quad (9)$$

Thus, we conclude that lasing can take place in a cavity-free system. In a toy-model, in which the cavity-free system consists of N_{at} atoms located at a single point, an analytical expression for the lasing threshold can be derived (see Appendix). It is important that the value of this threshold remains finite when the box size $L_U \to \infty$ (see Appendix). That is, the lasing in a cavity-free system does not depend on the box size.

4. Mechanism for cavity-free lasing

To understand the origin of lasing in a cavity-free ASE system, it is useful to look at the textbook picture of lasing in a system with a resonator from a nonconventional point of view. An open resonator placed in free space leads to a local maximum in the DOS at the resonance frequency [25]. The modes forming the maximum have almost the same frequencies and can interfere. A certain configuration of these modes interferes constructively, increasing the field intensity. Consequently, excited atoms are stimulated to emit photons into the configuration of the modes more intensively, increasing the field further. Thus, the DOS maximum serves as positive feedback providing coherent radiation of the system.

Below we show that the mechanism, in which a DOS maximum provides positive feedback, can be realized in a cavity-free system. The main difference between our mechanism and the mechanism of [25] is that in a cavity-free system, the DOS maximum is created not by a resonator but by the interaction of free-space modes with an active medium. This interaction results in pulling eigenfrequencies of modes having fixed wavenumber to the transition frequency.

Although the coherence threshold γ_{coh} exists in a noiseless system, the mechanism responsible for mode pulling becomes apparent by considering the system with quantum noise that is responsible for spontaneous emission. In such a system, the complex amplitude of the free-space mode with a fixed wavenumber, k, fluctuates. The spectrum of this amplitude defines the corresponding frequency. Since at the transition frequency of active atoms, the gain coefficient has a maximum, below γ_{coh} the closer modes to the transition frequency, the slower their decay [1,16]. Therefore, the longest-living fluctuations are those that pull the mode toward the transition frequency. Such fluctuations give the greatest contribution to the mean frequency. To demonstrate this frequency pulling, we consider the spectrum of the complex amplitude of each space harmonic. For this purpose, we calculate the values of the correlator

$$A_j(\tau) = \left\langle a_j^*(t_{st} + \tau) a_j(t_{st}) \right\rangle \Big/ \left\langle a_j^*(t_{st}) a_j(t_{st}) \right\rangle \quad (10)$$

and the spectrum of the j-th harmonic

$$S_j(\omega) = \mathrm{Re} \int_0^{\infty} A_j(\tau) \exp(i\omega\tau) d\tau, \quad (11)$$

in the stationary regime reached at $t = t_{st}$. Figure 2 shows the frequency spectra of the harmonic with the wavevector $k_b = 0.978\omega_{TLS}/c$ for pump rates below, near, and above the lasing threshold. Deep below the threshold, the spectrum has a pronounced maximum at $k_b = 0.978\omega_{TLS}/c$ and a noticeable high-k wing with a weak maximum at $k_{TLS} = \omega_{TLS}/c$. With increasing pumping, the first maximum decreases while the second maximum increases. Thus, the mean frequency of

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the space harmonics shifts toward the atomic transition. This maximum sharply grows with a further rate increase. We refer to this phenomenon as frequency pulling of space harmonics with different wavenumbers. The frequency of the space harmonic with the wavenumber k_j is defined as

$$\omega(k_j) = \int_0^\infty \omega S_j(\omega) d\omega / \int_0^\infty S_j(\omega) d\omega. \tag{12}$$

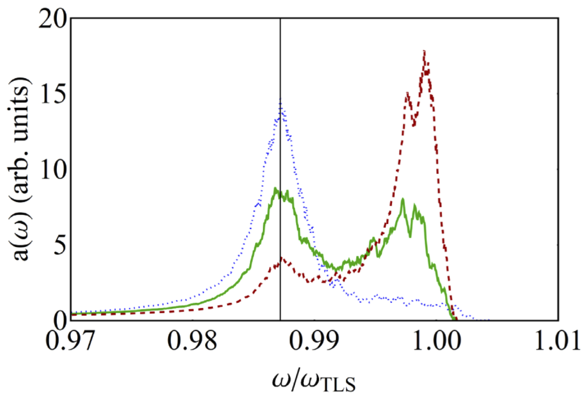

Fig. 2. Spectra of the free-space mode with the free-space eigenfrequency $\omega = 0.978\omega_{TLS}$ for different pump rates: $\gamma_P = \gamma_D$ (the dotted blue curve), $\gamma_P = 1.5\gamma_D$ (the solid green curve), and $\gamma_P = 2\gamma_D$ (the dashed red curve) for an extended system. Below the threshold, the maximum of the spectrum is at $\omega = 0.978\omega_{TLS}$ (marked by the vertical black line); with an increase in the pump rate, the maximum at the atom transition frequency grows. $G(N_c)L_{am} = 26$.

Figure 3 shows that near $k_{TLS} = \omega_{TLS}/c$, due to the frequency attraction of free-space modes to the transition frequency of active atoms, an interval of wavenumbers in which the dispersion curve tends to a horizontal line $\omega(k) = \omega_{TLS}$ arises. The size of this interval is determined by the level of noise and tends to infinity when noise vanishes. The flattening of the dispersion curve results in a decrease of the group velocity $v_{gr} = \partial\omega/\partial k$, which tends to zero, and a sharp increase

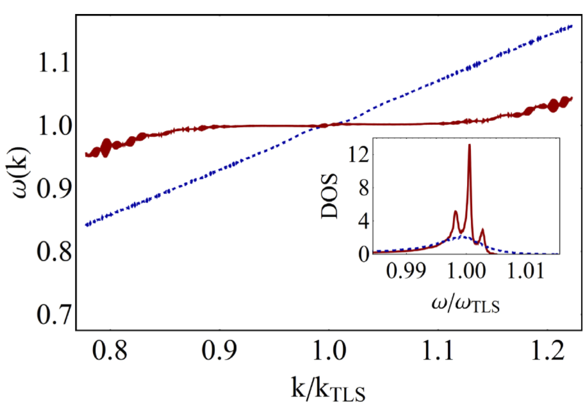

Fig. 3. The dependence of the mean frequency on the wavenumber for $\gamma_P = 2\gamma_D$ (the dashed blue curve) and $\gamma_P = 1000\gamma_D$ (the solid red curve). The DOSs near the transition frequency for $\gamma_P = 2\gamma_D$ (the dashed blue curve) and $\gamma_P = 1000\gamma_D$ (the solid red curve) are shown in the inset. The side maxima correspond to antisymmetric solutions, which have weaker interaction with atoms.

in the DOS, which is inversely proportional to v_{gr}. Such a behavior of the DOS is shown in the inset in Fig. 3.

$S_j(\omega)$ defined by Eq. (11) may be considered as a contribution of the j-th harmonic into the DOS dependence on the frequency, so that $\mathrm{DOS}(\omega) = \sum_j S_j(\omega)$ [19]. The inset in Fig. 3 shows that for $\gamma_P = 1000\gamma_D$, the DOS has a sharp maximum at the atomic transition frequency. The maximum of the DOS and the zero of the group velocity lead to lasing in the system (see also [17]).

5. Conclusion

We have demonstrated that for a high gain coefficient of active media, in a cavity-free ASE system, a new pump threshold - a coherence threshold, γ_{coh}, may arise. Above γ_{coh}, an ASE system generates coherent light. The cause for coherent radiation is pulling frequencies of free-space modes with different wavevectors toward the transition frequency of active atoms. This pulling occurs due to the interaction of free-space modes with active atoms. Consequently, a peak in the DOS of the system arises, and the group velocity of light decreases sharply. This decrease leads to an increase in the strength in the interaction between the EM field and the active medium [17], which ultimately leads to lasing.

Our analysis shows that the coherence threshold cannot be identified with the lasing threshold, which may arise due to reflection at the boundaries of an active medium and vacuum. To demonstrate this, we compare the lasing threshold in our system with the lasing threshold in a system with a Fabry-Perot resonator. The comparison shows that in order to obtain lasing at the coherence threshold due to feedback provided by the Fabry-Perot resonator, the amplitude reflection coefficient should be ≈0.079. This is much larger than the reflection from the boundaries of an active medium and vacuum, which in our system is $\approx 3.7 \cdot 10^{-3}$. Moreover, the lasing can take place even in a cavity-free toy-system consisting of N_{at} atoms located at one point (see Appendix). In this system, the length of the active medium is zero, and there is no Fabry-Perot resonator that is formed in the active medium. In our system, the only condition necessary for lasing is a maximum in the DOS.

Even though our calculations demonstrate a possibility of lasing in a cavity-free system, we note, however, that difficulties that may arise in experiment can complicate an unambiguous interpretation of the nature of coherence of the output ASE [9,26–28].

Appendix. Toy-model: cavity-free system having N_{at} atoms located at one point

To demonstrate that in our system, lasing does not occur due to a Fabry-Perot resonator formed by the boundaries of the active medium, we consider the case of an infinite number of the modes that interact with N_{at} atoms located at the point $x = 0$. The difference between the eigenfrequency of the j-th mode and the transition frequency of the atoms $\Delta_j = \omega_j - \omega_{TLS} = \Delta_0(j - 1/2)$, where $\Delta_0 = \pi c/L$ is the step between mode frequencies. We seek the self-oscillating solution at the atom transition frequency ω_{TLS}. In other words, we investigate the possibility of an existence of the solution for which frequencies of all modes are pulled toward the atomic transition frequency. To describe this system, Maxwell-Bloch equations for slowly varying amplitude, Eqs. (1)–(3), are modified as

$$\frac{d}{dt}a_j = (-i\Delta_j - \gamma_a/2)a_j - i\Omega_R N_{at}\sigma, \tag{13}$$

$$\frac{d}{dt}\sigma = -\gamma_{deph}\sigma/2 + i\,\Omega_R\,D\sum_j^{mode} a_j, \tag{14}$$

$$\frac{d}{dt}D = (\gamma_P - \gamma_D) - (\gamma_P - \gamma_D)D + 2i\Omega\left(\sigma\sum_j^{mode} a_j^* - \sigma^*\sum_j^{mode} a_j\right). \tag{15}$$

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From Eq. (13), the stationary value of the mode amplitude a_j may be expressed through the atom polarization σ as

$$a_j = \frac{-i\Omega_R N_{at}}{\gamma_a/2 + i\Delta_j}\sigma. \tag{16}$$

Inserting Eq. (16) into Eq. (14) one obtains

$$0 = -\gamma_{deph}\sigma/2 + \Omega_R^2 \sigma N_{at} D \sum_j^{mode} \frac{1}{\gamma_a/2 + i\Delta_j}. \tag{17}$$

From Eq. (17) we obtain the pumping threshold for the non-trivial solution:

$$D^{st} = D_{th} = \frac{\gamma_a\gamma_\sigma}{4\Omega_R^2 N_{at}} \frac{1}{\sum_j^{mode} \frac{1}{1+2i\Delta_j/\gamma_a}} \equiv \frac{\gamma_a\gamma_\sigma}{4\Omega_R^2 N_{at}\xi}, \tag{18}$$

where we denote

$$\xi \equiv \sum_j^{mode} \frac{1}{1+2i\Delta_j/\gamma_a}. \tag{19}$$

The main point here is that in the limit of an infinite number of modes, the threshold value of the solution with pulled mode frequencies, Eq. (18), tends to a finite value. Indeed, in this limit,

$$\xi = \sum_{j=-\infty}^{\infty} \frac{1}{1+2i\Delta_j/\gamma_a} = 2\sum_{j=1}^{\infty} \frac{1}{1+4\Delta_0^2(j-1/2)^2/\gamma_a^2} = \frac{\pi}{2\Delta_0/\gamma_a} \tanh\left(\frac{\pi}{2\Delta_0/\gamma_a}\right) \tag{20}$$

and

$$D_{th} = \frac{\Delta_0\gamma_\sigma}{2\pi\,\Omega_R^2 N_{at} \tanh(\pi\,\gamma_a/2\Delta_0)}. \tag{21}$$

When the size of the box tends to infinity, $L_U \to \infty$, the step between the mode frequencies tends to zero $\Delta_0 \to 0$ and Eq. (21) takes the form

$$D_{th} = \frac{\Delta_0\gamma_\sigma}{2\pi\,\Omega_R^2 N_{at}}. \tag{22}$$

Taking into account that with an increasing box size L_U, first, the frequency distance between the modes decreases as $\Delta_0 \sim L_U^{-1}$, and second, the coupling Rabi constant decreases as $\Omega_R \sim L_U^{-1/2}$, we obtain that the lasing threshold of the solution for which frequencies of all modes are pulled toward the atomic transition frequency does not depend on the system size. Moreover, this threshold does not depend on the relaxation rate of the EM field in the mode, γ_a, provided that $\gamma_a/\Delta_0 \gg 1$. The last condition is always true when $L_U \to \infty$ ($\Delta_0 \sim L_U^{-1}$) and γ_a is not zero.

It should be emphasized that the threshold for each separate mode, $D_{th}^{(0)} = \gamma_a\gamma_\sigma/4\Omega_R^2 N_{at}$, increases as $D_{th}^{(0)} \sim L_U$ with an increase in the box size L_U. Thus, the threshold for the solution, for which all mode frequencies are pulled toward the atomic transition frequency, is smaller than that for each mode, and this threshold is finite in the limit of an infinite system size.

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