

Scaling in one-dimensional localized absorbing systems

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(Received 18 January 2001; published 13 June 2001)

Numerical study of the scaling of transmission fluctuations in the one-dimensional localization problem in the presence of absorption is carried out. Violations of single-parameter scaling for lossy systems are found and explained on the basis of a new criterion for different types of scaling behavior derived by Deych *et al.* [Phys. Rev. Lett., **84**, 2678 (2000)].

DOI: 10.1103/PhysRevB.64.024201

PACS number(s): 71.55.Jv, 72.15.Rn, 42.25.Bs

I. INTRODUCTION

The single-parameter scaling (SPS) hypothesis is the cornerstone of the current understanding of the localization phenomena. It was originally formulated in terms of the scaling behavior of the conductance of disordered conductors,¹ where it was suggested that when the length, L , of a disordered conductor increases, the evolution of the conductance, $g(L)$, is determined by a single parameter, g itself. For one-dimensional (1D) systems the Landauer's formula² expresses the electron conductance in terms of electron reflection, $R(L)$, and transmission, $T(L)$, coefficients as $g(L) = T(L)/R(L)$, and thereby allows considering of electron transport on an equal footing with, for example, propagation of light. It was recognized later that SPS must be understood in terms of properties of the entire distribution of g (or T), and that the most appropriate quantity to deal with is³

$$\tilde{\gamma}(L) = (1/2L) \ln[1 + 1/g(L)] = (1/2L) \ln[1/T(L)].$$

In the limit of large L , this parameter is normally distributed with the average $\gamma = \langle \tilde{\gamma}(L) \rangle = \lim_{L \rightarrow \infty} \tilde{\gamma}(L)$ and the variance $\sigma^2(L) = \langle \tilde{\gamma}^2(L) \rangle - \gamma^2$. The limiting value of $\tilde{\gamma}(L)$ is known in the theory of products of random matrices⁴ as the Lyapunov exponent (LE). The inverse quantity, the localization length, $l_{loc} = \gamma^{-1}$, determines the main length scale in the localization regime.⁵ SPS in this context means that the variance of $\tilde{\gamma}$ is not an independent parameter, but it is determined by γ itself implying a simple relationship between two quantities:³

$$\sigma^2 = \gamma/L. \quad (1)$$

This expression was obtained for the one-dimensional model in Ref. 3 assuming complete randomization of the phases of complex transmission and reflection coefficients over a microscopic length scale $l_{ph} \ll l_{loc}$ (phase-randomization hypothesis). Later the phase-randomization hypothesis was used by many different authors to rederive Eq. (1) (see, for instance, Refs. 5,6), and the inequality $l_{ph} \ll l_{loc}$ came to be regarded as the criterion for SPS. However, there were earlier signs that the phase-randomization hypothesis is neither necessary nor sufficient condition for SPS to occur. For instance, numerical simulations of Ref. 6 and analytical calculations of Ref. 7 showed that in the center of a conductivity band of the 1D Anderson model, SPS holds even though the

phase is not randomized, provided that the disorder is weak. Later, numerical simulations of a random, periodic-on-average model⁸ demonstrated a strong violation of SPS in the band gaps of the spectrum of the underlined system without disorder, which existed even for weak disorder and, actually diminished, when disorder increased. This was contrary to the behavior found for states from the original conducting band, for which minor deviations of the variance from Eq. (1) occurred when disorder becomes strong enough.⁶

The final realization of the fact that the phase randomization hypothesis has nothing to do with SPS came in Ref. 9. In that paper, the variance of LE was calculated exactly for the Lloyd model and the SPS equation (1) was derived without *ad hoc* assumptions.¹⁰ It was found that the emergence of SPS is governed by a new length scale l_s , related to the integral density of states with the new criterion for SPS being $\kappa = l_{loc}/l_s \gg 1$. On the basis of the exact solutions it was conjectured⁹ that in the region of the spectrum close to its original boundary, the parameter l_s can be defined in a generic case as

$$l_s = a/N(E), \quad (2)$$

where a is the lattice constant and $N(E)$ is the number of states between the closest genuine boundary of the spectrum of the disordered system and E , normalized by the total number of states in the band [such that $0 < N(E) < 1$]. It follows from the established criterion that the violation of SPS occurs in the regions of the spectrum with the depleted differential density of states. It is well known⁵ that these regions correspond to fluctuation states arising outside the initial spectrum of the system. Therefore, the deviations from SPS must occur at the gap sides of initial boundaries of the spectrum as was observed in Ref. 8. Numerical studies undertaken in Ref. 9 evidenced that the criterion based upon the definition of l_s given by Eq. (2) was valid for the periodic-on-average model with rectangular distribution of random parameters. An additional implication of the results of Ref. 9 is that the random-matrix theory approach, which also reproduces Eq. (1),^{11,12} does not apply to spectral regions with depleted differential density of states.

The objective of this paper is to show that two different scaling regimes governed by the parameter l_s found in Ref. 9 exist also in disordered systems with absorption. Having in mind applications to light propagation in random photonic band-gap materials, we consider how the inclusion of small absorption affects scaling properties of transmission. We

show that the conjectured definition of l_s in terms of the integral density of states can also be applied to disordered systems with absorption.

Within the phase-randomization hypothesis approach, statistics of the transmission in lossy one-dimensional dielectrics was considered analytically in a number of papers.^{13,14} Following Ref. 13 the relation between the variance and the localization length in the presence of absorption can be presented in the form

$$\tau = \tau_0(\beta) = 1 + 2\beta e^{2\beta} Ei(-2\beta), \quad (3)$$

where $\tau = \sigma^2 L / \gamma$, $\beta = l_{loc} / l_a$, and l_a is the absorption length in the absence of randomness.

$$Ei(x) = \int_{-\infty}^x dt \exp(t)/t$$

is the exponential integral. In the absence of absorption, Eq. (3) reduces to the regular SPS form $\tau = 1$. In the two limiting cases Eq. (3) gives the asymptotes $\tau = 1 - 2\beta \ln(1/\beta)$, $\beta \ll 1$ and $\tau = 1/2\beta$, $\beta \gg 1$.

Presuming that the length scale l_s retains its meaning in the case under consideration, we expect that the parameter τ deviates from the phase-randomization hypothesis prediction $\tau_0(\beta)$ [Eq. (3)] in the vicinity of the boundaries of the spectrum in accordance with the same criterion $l_{loc} \gg l_s$ as in Ref. 9. Using numerical simulations of a periodic-on-average one-dimensional lossy system, we show that, indeed, the parameter $\kappa = l_{loc} / l_s$ sets a valid criterion for validity of Eq. (3). As an additional benefit, we demonstrate that within the range of its validity, Eq. (3) represents a universal, model independent relation between the variance and the mean value of LE. The deviations from the phase-randomization-based results of Ref. 13 studied in our paper must be clearly distinguished from results of Ref. 15. In the latter paper,

break down of the phase randomization was obtained in the case of very strong disorder and strong absorption for states at the center of the original band. The results of our paper indicate that (i) violation of the generalized single-parameter scaling occurs at weak disorders for states close to the band edge of the original spectrum, (ii) this violation is not related to the phase randomization but is controlled by the parameter l_s .

II. MODEL AND THE METHOD OF CALCULATIONS

We consider a classical transverse electromagnetic wave propagating normally through a stack of alternating dielectric slabs with dielectric constants ϵ_1 and ϵ_2 . The widths of the stacks of the first kind is distributed uniformly in the interval $(d_1 - \delta, d_1 + \delta)$ while the width of the others is being kept constant, d_2 . The propagation of the waves in the superlattice consists of free propagation in the slabs and scattering at the interfaces, where the boundary conditions should be satisfied. It can be described using the transfer-matrix formalism for the vector $v_n = (E_n, E'_n/k)$, where E_n, E'_n are the electric field and its derivative at n th interface and $k = \omega/c$. The presence of absorption can be accounted for by adding a constant complex part to the dielectric functions: $\epsilon_1 = \epsilon_1^{(0)}(1 + i\alpha)$ and $\epsilon_2 = \epsilon_2^{(0)}(1 + i\alpha)$, where α is a damping coefficient. Vectors on neighboring interfaces are connected via the transfer matrix

$$\hat{T}_n = \begin{pmatrix} \cos k_n d_n & (1/k_n) \sin k_n d_n \\ -k_n \sin k_n d_n & \cos k_n d_n \end{pmatrix}, \quad (4)$$

where $k_n = k \sqrt{\epsilon_n}$. The transfer matrix of the entire system is $\hat{\mathbf{T}}(\alpha, L) = \prod_{n=1}^{2N} \hat{T}_n(\alpha)$, where $L = N(d_1 + \langle d_2 \rangle)$. LE is defined through the transmission coefficient for the superlattice:

$$\gamma(L, \alpha) = -\frac{1}{2L} \langle \ln T(\alpha, L) \rangle = -\frac{1}{2L} \left\langle \ln \left| \frac{2 \det \hat{\mathbf{T}}(\alpha, L)}{[\hat{\mathbf{T}}_{11}(\alpha, L) + \hat{\mathbf{T}}_{22}(\alpha, L)] - i[\hat{\mathbf{T}}_{12}(\alpha, L) - \hat{\mathbf{T}}_{21}(\alpha, L)]} \right|^2 \right\rangle, \quad (5)$$

here $\langle \dots \rangle$ denotes the average over an ensemble of configurations. We find that using this definition in numerical simulations has one significant shortcoming. In long systems, the transmission coefficient T falls below the computer roundoff accuracy. The usual remedy for this problem in the absence of absorption is to use an alternative definition of γ ,

$$\gamma(L) = \frac{1}{2L} \left\langle \ln \frac{||\hat{\mathbf{T}}(L)\mathbf{v}_0||^2}{||\mathbf{v}_0||^2} \right\rangle \quad (6)$$

(\mathbf{v}_0 is a generic vector), which allows one to consider very long systems.⁴ In the presence of absorption, this definition must be generalized because the simple substitution of the transfer matrix, Eq. (4), in this equation would lead to a wrong result. The problem is that an exponentially growing

eigenvalue of the transfer matrix appears in the denominator of Eq. (5), while in Eq. (6) it is in the numerator. Therefore, the contribution from absorption enters the final answer for the LE with different relative signs in these two equations. We argue that Eq. (6) must be modified in order to agree with the original definition of γ as follows:

$$\gamma(L, \alpha) = \frac{1}{2L} \left\langle \ln \frac{||\hat{\mathbf{T}}(-\alpha, L)\mathbf{v}_0||^2}{||\mathbf{v}_0||^2} \right\rangle. \quad (7)$$

To show the equivalency of Eqs. (5) and (7) we, first, diagonalize the transfer matrix $\hat{\mathbf{T}}(\alpha, L)$:

$$\hat{\mathbf{T}}(\alpha, L) = \hat{U}^\dagger \hat{\mathbf{T}}^{(D)}(\alpha, L) \hat{U} = \hat{U}^\dagger \begin{pmatrix} e^{v_1(\alpha)2N} & 0 \\ 0 & e^{v_2(\alpha)2N} \end{pmatrix} \hat{U}, \quad (8)$$

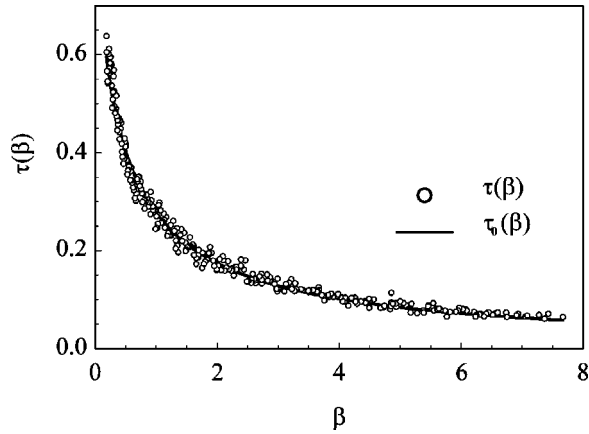


FIG. 1. Open-circles depict the parameter $\tau(\beta)$ computed for the periodic-on-average system of 100 000 layers far from the band edge. $0.86 < k < 1.4$, the disorder parameter is $\delta=0.45$, the damping $\alpha=0.00125$. The solid line shows $\tau_0(\beta)$ given by Eq. (3).

here \hat{U} and \hat{U}^\dagger are some unitary matrices. For argument sake, we assume that $|\nu_1(\alpha)| \geq |\nu_2(\alpha)|$. Next, we notice the following relation between eigenvalues of the transfer matrix

$$\nu_1(\alpha) = -\nu_2(-\alpha), \quad (9)$$

which follows simply from the reciprocity of our system. Indeed, propagation of the waves in the opposite-direction should be described by the matrix $\hat{\mathbf{T}}'(\alpha, L) = [\hat{\mathbf{T}}(-\alpha, L)]^{-1}$, that leads to Eq. (9). Now, substituting $\hat{\mathbf{T}}^{(D)}(\alpha, L)$ into Eqs. (5) and (7) (corrections due to matrices \hat{U} and \hat{U}^\dagger are negligible in the limit $N \rightarrow \infty$) one obtains $\gamma(L, \alpha) = -\nu_2(\alpha)$ and $\gamma(L, \alpha) = \nu_1(\alpha)$, respectively. Along with Eq. (9) this shows that these equations lead to the same value of LE. To make sure that the statistics of $\gamma(L, \alpha)$ given by both equations are also the same, we calculated numerically distribution functions for $\gamma(L, \alpha)$ using both Eqs. (5) and (7) for relatively short systems [such that Eq. (5) is still applicable] and found that they are identical.

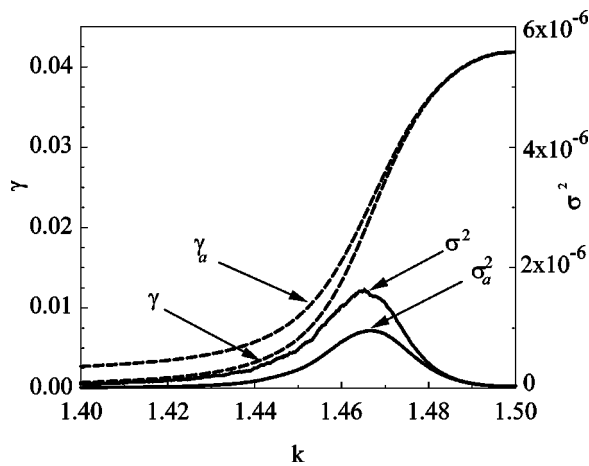


FIG. 2. The Lyapunov exponent γ and its variance σ in the vicinity of the first band gap is plotted against the frequency. The disorder parameter is $\delta=0.25$, absorption is $\alpha=0.0025$.

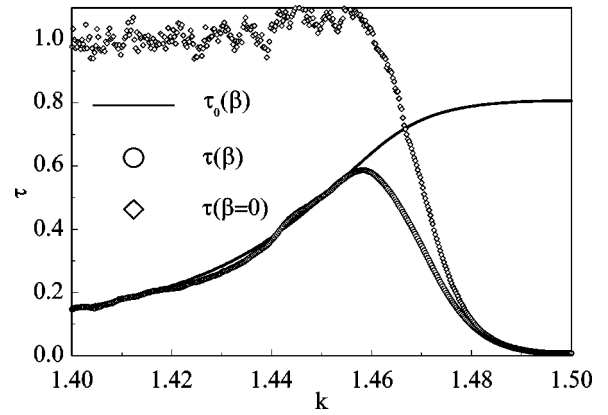


FIG. 3. $\tau(0)$ (rhombuses), $\tau(\beta)$ (open circles), and $\tau_0(\beta)$ (solid line) on the band edge plotted as a function of frequency for the same set of parameters as in Fig. 2.

III. RESULTS

In numerical simulations we used the following set of parameters $\epsilon_1=1.2$, $\epsilon_2=1$, and $\langle d_1 \rangle = d_2 = 1$. The disorder parameter, δ , and the absorption rate, α , were variable parameters. To calculate the moments of γ , we averaged the characteristics of systems as long as 100 000 layers over 5000 realizations. The size of the stack was chosen to be at least five times the localization length or the absorption length. In the ordered system the first forbidden gap lies between $k=1.456$ and $k=1.543$. Since the localization length depends on the frequency of the wave, it is possible to study the function $\tau(\beta(\omega))$ by changing the frequency. First, we compare our numerical results with the analytical formula of Ref. 13 in the region of frequencies well inside the first allowed band, where we expect these results to coincide. Figure 1 shows excellent agreement between the computed $\tau(\beta)$ for $0.86 < k < 1.4$ and Eq. (3).

The localization length decreases rapidly when the frequency approaches the band edge, while LE, consequently, increases. As follows from Eq. (1), in the absence of absorption, σ^2 should follow the LE. Figure 2 depicts the depen-

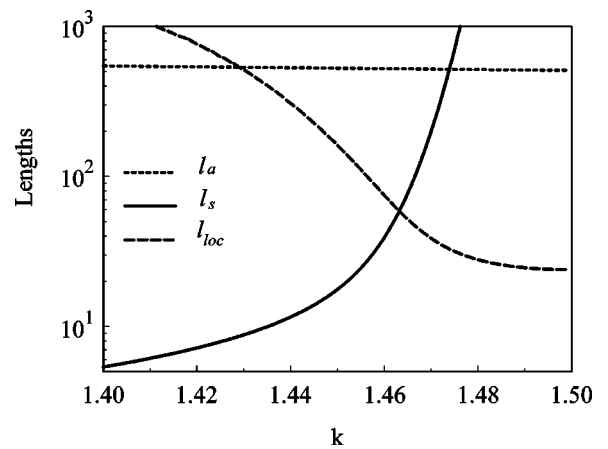


FIG. 4. Length scales for the system with parameters used in Fig. 2. The solid line depicts l_s , the dashed line is the localization length, l_{loc} , and the dotted line is the absorption length, l_a .

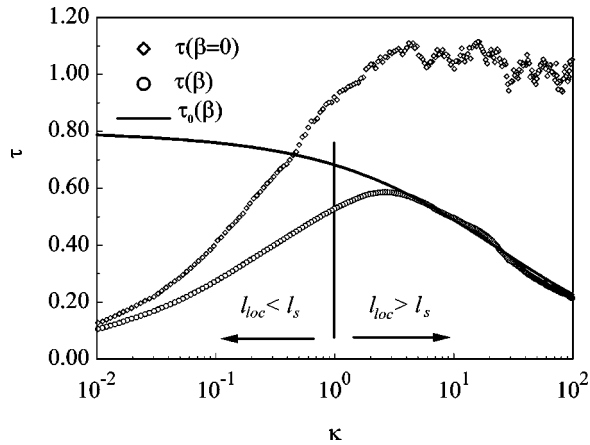


FIG. 5. $\tau(0)$ (rhombuses), $\tau(\beta)$ (open circles), and $\tau_0(\beta)$ (solid line) on the band edge plotted as a function of $\kappa = l_{loc}/l_s$ for the same set of parameters as in Fig. 2.

dence of LE and its variance with and without absorption. As one can see, at the band edge, $k = 1.456$, SPS breaks down. Indeed, while LE grows with the increasing frequency, the variance drops in both cases, with and without absorption.

In order to compare numerical and analytical results in the presence of absorption, we calculated the parameter τ as a function of frequency k , using the data presented in Fig. 2. The results are presented in Fig. 3, from which one can see that the computed $\tau(\beta)$ deviates from $\tau_0(\beta)$ that represents Eq. (3), and that this deviation occurs at the same frequency at which $\tau(0)$ deviates from unity. These graphs convincingly demonstrate, that even in the presence of absorption, the spectrum of the system is separated in groups with different scaling properties, and that the boundary between the groups coincide with the boundary of the original spectrum.

The next question we need to address is whether the transition between the scaling regimes is governed by the same parameter l_s that was introduced in Ref. 9. According to that paper, the new parameter l_s becomes greater than the localization length at the band edge, which results in a deviation from SPS. Using definition of l_s suggested in Ref. 9 regarding the integrated density of states, in Eq. (2), we numerically calculated this parameter for the system studied in the present paper. The density of states was calculated using the phase formalism (see, for instance, Ref. 5) for the system without absorption. Figure 4 shows all relevant length parameters: localization length, absorption length, and l_s .

At the band edge, l_s grows rapidly because very few new states appear within the former band gap, and $N(E)$ must already be close to unity at the band edge. It reaches unity at a new fluctuation boundary of the spectrum near the center of the gap. We assume that the disorder is not very strong such that the fluctuation boundaries inside former band gaps exist. If the disorder is strong enough, or if its statistical properties are such that the entire band gap is filled with fluctuation states a definition of l_s is still possible, but the situation becomes more complicated, and we do not consider it here. At certain points l_s grows larger than the localization length l_{loc} , and one can find comparing Figs. 3 and 4, that $\tau(\beta)$ starts deviating from $\tau_0(\beta)$ at the same frequency. In

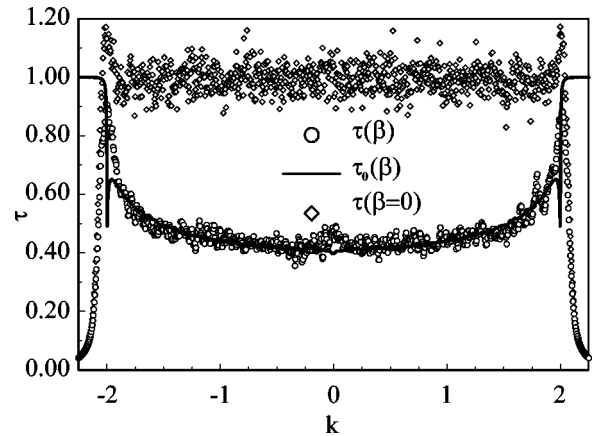


FIG. 6. $\tau(0)$ (rhombuses), $\tau(\beta)$ (open circles), and $\tau_0(\beta)$ (solid line) plotted as a function of frequency averaged over 1000 realizations. The length of the system 10 000 cells, disorder parameter $\delta = 0.25$, damping $\alpha = 0.0025$.

order to make this more transparent, we plot various τ 's versus $\kappa = l_{loc}/l_s$ in Fig. 5.

Without absorption, we see the crossover to SPS in its pure form at $\kappa \approx 1$ reported in Refs. 8 and 9. When absorption is present, the crossover still occurs at $\kappa \approx 1$, but now to the modified SPS behavior $\tau_0(\beta)$. We stress that the crossover occurs while $a \ll l_{loc} \ll L$ and $a \ll l_a \ll L$ so that the system remains in the meaningful scaling regime. This demonstrates that the condition $1 \ll \kappa$ establishes the criterion for modified SPS even in the presence of the absorption.

To check that our results are not model specific we also studied the Anderson model, with absorption introduced as a nonrandom imaginary part of the on-site random energy. In the Anderson model without disorder and absorption, there is a single allowed band from $E = -2$ to $E = 2$.

Figure 6 shows that $\tau(\beta)$ deviates from $\tau_0(\beta)$ at the edges of the allowed band as would be expected on the basis of the results presented for the periodic-on-average system. Inside the conductivity band, numerical and analytical results show excellent agreement. This is interesting in itself, since Eq. (3) was derived for a continuous model,¹³ and we see that in the SPS regime it holds also for the tight-binding discrete model.

In conclusion, we computed the Lyapunov exponent and its variance for a periodic-on-average layered system and for the one-dimensional Anderson model. We studied the deviation from the absorption-modified SPS expression for the variance.¹³ We showed that the new length scale, introduced in Ref. 9 in order to explain violation of SPS in the systems without absorption, retains its significance when absorption is present. We also showed that the same criterion $\kappa = l_{loc}/l_s \gg 1$ derived for lossless systems in Ref. 9 controls scaling behavior in lossy systems.

ACKNOWLEDGMENTS

We are indebted to A. Genack for reading and commenting on the manuscript. This work was partially supported by NATO Linkage Grant No. N974573, CUNY Collaborative Grant, and PSC-CUNY Research Award.

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