Delay-time statistics for diffuse waves

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(Received 28 January 1999)

We formulate a theory for the statistics of the dynamics of a classical wave propagating in random media by analyzing the frequency derivative of the phase under the assumption of a Gaussian process. We calculate frequency correlations and probability distribution functions of dynamical quantities, as well as the first non-Gaussian C 2 correction. In A. Z. Genack, P. Sebbah, M. Stoytchev, and B. A. van Tiggeleen, Phys. Rev. Lett. 82, 715 (1999), microwave measurements have been performed to which this theory applies.

PACS number(s): 42.25.−p

I. SCOPE

Fluctuations in wave propagation in disordered media are extensively treated in standard references [2,3]. Perhaps the best known feature is the apparently Gaussian statistics of the complex wave field, with real and imaginary parts as independent variables. This results in the familiar Rayleigh statistics \( P(I) \sim \exp(-I \langle I \rangle) \) of the intensity \( I \). The Gaussian process is a natural consequence of the central limit theorem if one assumes that the complex field results from a coherent superposition of many independent wave trajectories.

Modern speckle theory, founded in the 1980s and still developing rapidly, changes this simple picture considerably [3,4]. New features in speckle statistics have been observed such as short- and long-range intensity correlations, e.g., in frequency [5] or time [6], the memory effect [7], universal fluctuations [8,9], and non-Rayleigh statistics [10] for the intensity, many of these explained in terms of non-Gaussian field statistics. An old tool from nuclear physics, random matrix theory (RMT), has been successfully introduced in this field and seems to capture many “universal” aspects [11]. Only recently, RMT has solved the fundamental problem of the statistical properties of the so-called phase delay times—which for classical waves are the derivatives of the phase shifts of the \( S \) matrix with respect to frequency \( d \phi/d \omega \)—at least for chaotic billiards where Dyson circular ensembles are known to apply [12,13]. The applicability of the Dyson RMT in disordered systems has been estimated to hold for energies small compared to the Thouless energy [14]. Pioneering work by Dorokhov [15] and Mello, Pereyra, and Kumar [16] extended RMT to disordered wires. The extension of DMPK theory towards dynamical problems is under active investigation [17,18].

The popularity of the phase derivative \( d \phi/d \omega \) stems from its interpretation as a time delay in scattering [19]. This quantity describes the dynamics of a very narrow-band wave packet. In homogeneous media it relates directly to the group velocity \( v_G \) [20] whereas in random media its ensemble average is inversely proportional the transport velocity \( v_E \) figuring in the diffusion constant of the average intensity [22].

Unlike \( v_E \), however, \( d \phi/d \omega \) can be considered for arbitrary realizations. The restriction to narrow-band wave packets introduces sometimes apparently interpretational problems, such as negative group velocities or negative delay times, that have been extensively studied in literature [21].

The phase delay time relates directly to a fundamental dynamic quantity in condensed matter, namely the number of states per frequency interval \( d \omega \) inside the scattering medium \( N(\omega) \) [23],

\[
\frac{1}{\pi} \sum_{a,b} I_{ab} \frac{d \phi_{ab}}{d \omega} = N(\omega),
\]

where \( \sqrt{I_{ab}} \exp(i \phi_{ab}) = I_{ab} \) is the complex transition amplitude from mode \( a \) to \( b \). Unlike the summation in the Landauer formula for conductance, the summation runs over the \( M \) channels in both reflection and transmission. Equation (1) is a manifestation of Friedel’s theorem [24], originally devised for screening problems in the solid state, though with elegant applications to many scattering problems [25–27], including ones in RMT [28]. Alt’shuler and Shklovskii [14] demonstrated the central role of the statistics of \( N(\omega) \) in the understanding of the relation between level repulsion—described as a concept in RMT—and universal conductance fluctuations, a basic element in modern speckle theory, a relation confirmed numerically [17]. It is crucial that the density of states in an open (scattering) medium is ill-defined due to the finite (Thouless) width of the levels. As shown in Ref. [23], the left-hand side of Eq. (1) is well-defined and proportional to the integral \( \int d\mathbf{r} |\psi_0(\mathbf{r})|^2 \) over the sample, which is—within the original Friedel argument—recognized as the “stored charge.” For light it equals the stored electromagnetic energy [27].

Though not free from controversy, Eq. (1) calls for the interpretation of \( W_{ab} = I_{ab} d \phi_{ab}/d \omega \) as the weighted delay time for a transition from channel \( a \) to \( b \) [29], to be distinguished from the “proper” delay times defined as the eigenvalues of the Wigner-Smith matrix \( \mathbf{Q} = -iS^* \cdot \partial S/\partial \omega \), and its trace \( \text{Tr} \mathbf{Q} = \Sigma_{a,b} W_{ab} = \pi N(\omega) \), called the Heisenberg
time $\tau_H$. The channel average $\tau_H/2M$ is associated with the Wigner-Smith phase delay time $\tau_w$. Finally, one could call $d\phi_{ab}/d\omega$ the single channel delay time for a transition from channel $a$ to $b$, irrespective the transition probability $I_{ab}$.

The intention of this work is to formulate a statistical theory for the dynamical matrix elements $d\phi_{ab}/d\omega$ and $W_{ab}$ for diffuse waves, using concepts developed for the static cross section $I_{ab}$ [4]. This theory can be applied equally well to phenomena involving phase variations with other variables. The choice of frequency is stimulated by microwave experiments [1,30]. We shall adopt the “$C_1$ approximation,” which is known to work best for the static “one channel in—one channel out” matrix element $I_{ab}$ (provided the conductance $g = M/1 > 1$), but which as far as we know has not been worked out for dynamic quantities such as $d\phi_{ab}/d\omega$ and $W_{ab}$. The summation $\Sigma_b W_{ab}$ equals the diagonal element $Q_{ab}$ of the Wigner-Smith matrix and may, like the total transmission $\Sigma_b T_{ab}$ from channel $a$, be subject to $C_2$ correlations [5]. It is known that even a Gaussian process contains phase correlations between speckle spots [31]. The last section of this work will address the first results for $C_2$ frequency correlations in the dynamic matrix element $W_{ab}$. As yet, we have not been able to formulate a non-Gaussian theory for $d\phi_{ab}/d\omega$. We note, however, that experiments have shown this quantity to be highly Gaussian [1].

II. GAUSSIAN APPROXIMATION

The $C_1$ approximation is equivalent to the assumption of a circular complex Gaussian process [2] of the complex set $t_{ab}$. A circular process for $K$ complex field amplitudes $E_i(b) = t_{ab}(i)E_i(a)$ requires that $E_1 = 0$ and that $E_i E_j^*$ = 0. The index $i$ here labels $K$ different frequencies for a given channel transition $ab$. The joint distribution is given by

$$P(E_1, E_2, ..., E_K) = \frac{1}{\pi^K \det C} \exp\left(-\sum_{i,j=1}^{K} E_i C_{i,j} E_j^*\right),$$

where $C_{ij} = \langle E_i E_j^* \rangle$ is the Hermitian variance matrix. We shall normalize $\langle E_j E_j^* \rangle = 1$ for all $j$, assuming $I_j = 1$ to be independent of $j$. For small frequency difference $\omega_1 - \omega_2 = \omega$, we can make the expansion $C_{12} = 1 + i\omega\omega + b \omega^2 + O(\omega^3)$, where $a$ and $b$ can be calculated from diffusion theory [4,32], which involves the diffusion constant $D$, sample length $L$, absorption length $L_a$, and transport velocity $v_T$. The latter contains the Wigner delay time of the scattering objects [22] and thus forms the crucial link between microdynamics and macrodynamics.

Probability distributions can be derived for $K = 2$ using a change of variables $E_j = A_j \exp(i\phi_j)$. As $\omega \to 0$, the stochastic variables can be chosen as $I = A_1^2$, $\phi' = d\phi_{ab}/d\omega$, $R = d\ln A_0/\omega_d$, and $\phi_{ab}$. Integrating out the phase shift $\phi_{ab}$ yields for the joint distribution,

$$P(\phi', R) = \frac{1}{\pi Q a^2} \exp(-I) \times \exp\left[-\frac{I}{Q a^2} (\phi' - a)^2 - \frac{I}{Q a^2} R^2\right].$$

The distribution functionally depends on a single parameter $Q = -2b/a^2 - 1 > 0$, which is shown in Fig. 1 as a function of absorption $L/L_a$. From the diffusion formula for $C(\Omega)$ in transmission used in Ref. [4], it follows that

$$Q = \frac{X^2 - 2 \sinh^2 \left(\frac{1}{2} \right) X \sinh 2X}{(X \cosh X - \sinh X)^2}.$$  

with $X = L/L_a$. In the absence of absorption, the mean delay time $\langle \phi' \rangle = a$ equals the diffuse traversal time $L^2/6D$ for waves in transmission, and $4L/Dv_T$ in reflection. In Fig. 1 we see how $\langle \phi' \rangle$ decays as absorption comes in. Measuring $\langle \phi' \rangle$ in reflection and transmission would give access to both transport mean free path $\ell = 3D/v_T$ and transport velocity $v_T$.

Even in the Gaussian approximation, $I$ and $\phi'$ are correlated observables. For constant intensity $I$ the delay time is normally distributed with spread $\Delta \phi' / \langle \phi' \rangle = \sqrt{Q/27}$. This has been confirmed experimentally [1]. The phase in dark points becomes ill-defined [31], which causes strong fluctuations in $\phi'$ at low intensities. Upon integration over $I$ and $R$ one finds

$$P\left(\phi' = \frac{\phi'}{\langle \phi' \rangle}\right) = \frac{1}{2} \frac{Q}{(Q + (\phi' - 1)^2)^{3/2}}.$$  

This algebraic law agrees with experimental data for microwaves in transmission [1] over seven orders of magnitude. It has the property that $\langle (\phi')^2 \rangle = \infty$, though any finite frequency grid $\Delta \omega$ transforms this divergence into a finite value $-\ln \Delta \omega$. From Eq. (3), we find for the distribution of the dynamic matrix element $W_{ab}$.
\[
P\left(\hat{W} = \frac{W}{\langle W \rangle}\right) = \frac{1}{\sqrt{\Omega + 1}} \exp\left(-\frac{2|\hat{W}|}{\theta(\hat{W}) + \sqrt{\Omega + 1}}\right).
\]

The Heaviside function \(\theta(x)\) vanishes for \(x<0\) and equals 1 for \(x>0\). The average \(\langle W \rangle = (I_1(\phi') - L/2MV_E)\) both in transmission and reflection. Like the intensity \(I\), \(W\) has an exponential distribution but unlike \(I\) it can take negative values. Though less probable, the existence of negative “delay” times is an interesting feature that is also observed in experiments [1] and is allowed by scattering theory [19]. By Eq. (3) these are most probable in “dark spots.” In transmission from a thick slab without absorption \(Q = 2/5\) and positive values for \(W\) are 12 times more probable than negative values. In reflection [22], \(Q \approx (3L/\pi)^2 > 1\), implying nearly equal probabilities for positive and negative \(W_{ab}\).

III. GAUSSIAN THEORY
FOR FREQUENCY CORRELATIONS

Correlation functions at two close frequencies provide a sensitive test of the validity of the Gaussian approximation in the experiment. Frequency correlations of \(\phi_{ab}^\prime\) and \(W_{ab}\) can be obtained from Eq. (2) with \(K = 4\) at the frequencies \(\nu \pm \omega/2\pm \Omega/2\) in the limit \(\omega \to 0\). The correlation matrix we need to study is

\[
C(\omega, \Omega) = \begin{pmatrix}
1 & C(\omega) & C(\Omega) & C(\Omega + \omega) \\
C(\omega) & 1 & C(\Omega - \omega) & C(\Omega) \\
C(\Omega) & C(\Omega - \omega) & 1 & C(\omega) \\
C(\Omega + \omega) & C(\Omega) & C(\omega) & 1
\end{pmatrix}.
\]

For \(\omega = 0\) this matrix contains a doubly degenerated eigenvalue \(\lambda_{1,2} = 0\) and two eigenvalues \(\lambda_{3,4}(\Omega) = 2 \pm 2|C(\Omega)|\). For \(\omega \to 0\) one gets

\[
\frac{\lambda_{1,2}(\Omega)}{\omega^2} = \xi_{1,2}(\Omega) = \frac{Qa^2}{2} + \frac{1}{2} \frac{|C'(\Omega) - iaC(\Omega)|^2}{1 - |C(\Omega)|^2} + \frac{1}{2} \frac{|z(\Omega)|^2}{1 - |C(\Omega)|^2}.
\]

with 
\(z(\Omega) = C''(\Omega)[1 - |C(\Omega)|^2] - a^2C''(\Omega) + 2iaC'(\Omega) + C(\Omega)C'(\Omega)^2\). The corresponding four eigenfunctions can be derived straightforwardly, among which the first two will be required to order \(\omega\), as customary in second-order perturbation theory. A careful analysis for \(\omega \to 0\) in the subspace spanned by the first two eigenvectors shows that the joint distribution for the four complex fields can be transformed into

\[
P(A_1, A_2, A_3, A_4, \phi_1, \phi_2, \phi_3, \phi_4) = \frac{1}{4\pi^2 \xi_1^2 \xi_2 \lambda_3 \lambda_4} \exp\left(-\frac{1}{4\xi_1}|A_1 - \bar{\mu} - iA_1 \phi_1 + (A_2 + \mu - A_3 + iA_3 \phi_3)\exp(i\phi_{13} - i\rho)|^2\right) \\
\times \exp\left(-\frac{1}{4\xi_2}|A_2 - \bar{\mu} + iA_1 \phi_1 - (A_2 + \mu - A_3 + iA_3 \phi_3)\exp(i\phi_{13} - i\rho)|^2\right) \\
\times \exp\left(-\frac{1}{\lambda_3}|A_3 \exp(i\phi_{13} + i\tau)|^2\right) \times \exp\left(-\frac{1}{\lambda_4}|A_4 - A_3 \exp(i\phi_{13} + i\tau)|^2\right),
\]

where \(\rho(\Omega)\) and \(\tau(\Omega)\) are the complex phases of \(z(\Omega)\) and \(C(\Omega)\),

\[
\mu^\pm(\Omega) = \frac{\gamma_1(\Omega) \pm \gamma_2(\Omega)\exp[i\rho(\Omega)]}{1 - |C(\Omega)|^2},
\]

and \(\gamma_1(\Omega) = C(\Omega)\bar{C}'(\Omega) + ia\), \(\gamma_2(\Omega) = iaC(\Omega) - C''(\Omega)\).

Equation (9) implies many correlations between amplitudes, phase, and phase derivatives, only two of which we shall discuss. We remark that it is straightforward to integrate out the amplitude derivatives \(A_{1,3}\).

A. \(d\phi/d\omega\) frequency correlation

The calculation of the normalized frequency correlation function of \(W_{ab} = I_{ab}d\phi_{ab}/d\omega\) involves straightforward integrations of Eq. (9) that can all be done analytically.

\[
\langle \hat{W}_{ab}(\nu - \Omega/2)\hat{W}_{ab}(\nu + \Omega/2) \rangle_c = \frac{1}{2a^2} \left[|C'(\Omega)|^2 - \text{Re} C(\Omega)C''(\Omega)\right] = F_1(\Omega).
\]

This result can be obtained more easily using a method that will be introduced later to calculate long-range correlations,
using the Gaussian rules to evaluate a correlation involving four fields.

The variance \((\Delta \hat{W}_{ab})^2 = F_1(0)\) has been plotted in Fig. 1 as a function of absorption and in transmission. Without absorption, the diffusion approximation for \(C(\Omega)\) in transmission [4] predicts \(F_1(0) = 6/5\). For large \(\Omega\) one can show that \(F_1(\Omega) \approx \frac{1}{2} \exp(-\sqrt{2\Omega L^2/D})\).

\[
\left| d\Phi \frac{d\phi}{d\omega}(\nu - \Omega/2)_{ab} d\phi \frac{d\phi}{d\omega}(\nu + \Omega/2)_{ab} \right| = \frac{1}{4\pi^2} \int_{-\omega}^{\omega} d\phi (\phi^2 - |\phi|) - 2 Re(e^{-i\phi} \Phi(\phi)) \\
+ \frac{H_2(R_{\phi})(\gamma_1 + \gamma_2 e^{i\phi}) - 2 Im \gamma_1 Im \gamma_2 \Phi(\phi)}{1 - R^2_{\phi}},
\]

where \(R_{\phi} = Re(C(\Omega)e^{i\phi})\). We have introduced the functions \(\Phi(\phi) = \arctan(\sqrt{(1 + x)/(1 - x)})/\sqrt{1 - x^2}\), \(H_1 = 2H_0 + x\), and \(H_2 = 2\lambda H_0 + 1\).

As \(\Omega \to 0\), Eq. (12) predicts a logarithmic divergence, not encountered in intensity correlation functions. For large frequency shifts, we find that

\[
C_\phi(\Omega) = |C(\Omega)|^2 (|C| \ll 1),
\]

i.e., \(C_\phi\) becomes identical to the \(C_1\) intensity correlation function which decays exponentially. In Ref. [1] the correlation function (12) has been compared to experiment, in a regime where the intensity is known to be subject to large non-Gaussian fluctuations. The excellent agreement between Eqs. (5) and (12) and experimental data seems to exclude the existence of non-Gaussian (long-range) correlations in the single channel phase delay time \(d\Phi/d\omega\). At present we have no easy explanation for this phenomenon. In Fig. 2 we show the correlation function \(C_\phi(\Omega)\) for different absorption lengths.

The \(\Omega\) integral of the phase-delay correlation \(C_\phi(\Omega)\) is required for the variance \(\Delta \phi\) of the cumulative phase. The latter is defined as

\[
\phi_{ab}(\nu) = \int_{\nu_0}^{\nu} d\omega \frac{d\phi_{ab}}{d\omega}(\omega)
\]

with respect to some reference frequency \(\nu_0\). The short-range frequency correlation of \(d\phi/d\omega\) ensures that to a good approximation \(\phi(\omega)\) is Gaussian distributed with a variance proportional to its average [30]. The relation \(\Delta \phi(\omega)^2 = K\langle \phi(\omega) \rangle\) has been verified experimentally for several frequencies [30], where \(K\) was seen to be more or less independent of frequency. It is easy to show that

\[
K = \left\langle \frac{d\phi}{d\omega} \right\rangle \int_{-\infty}^{\infty} d\Omega C_\phi(\Omega).
\]

B. \(d\phi/d\omega\) frequency correlation

The derivation of the frequency correlation function for the single channel delay time \(\phi_{ab}(\nu)\) involves more work. After analytically integrating out the amplitude derivatives \(A_1\), an integral over the phase shift \(\Phi_1\) remains that must be done numerically. Equation (9) finally yields

\[
K \text{ is shown in Fig. 1 as a function of absorption. The value } K = 1.0 \text{ near } L/L_a = 5 \text{ coincides with the experimental value reported in Ref. [30]. Without absorption we find the somewhat larger value } K = 1.61.
\]

IV. NON-GAUSSIAN FREQUENCY CORRELATIONS IN \(I_d \phi/d\omega\)

Consider the frequency correlation function of four field and introduce

\[
\Delta(\nu, \Omega, \Phi) = \langle E_{\nu - \Omega/2 - \omega} E_{\nu - \Omega/2 + \omega} E_{\nu} E_{\nu + \omega/2} E_{\nu + \omega/2} E_{\nu + \omega/2} \rangle - \langle E_{\nu - \Omega/2 - \omega} E_{\nu - \Omega/2 + \omega} E_{\nu} E_{\nu + \omega/2} E_{\nu + \omega/2} E_{\nu + \omega/2} \rangle.
\]

FIG. 2. Correlation function with frequency of the phase delay time \(d\phi/d\omega\), on the basis of Gaussian theory. Different values for the absorption \(L/L_a\) have been considered. The frequency has been scaled with the ensemble average delay time.
The following identity is straightforward to prove:

$$\langle W_{v-\Omega/2}W_{v+\Omega/2} \rangle = \lim_{\omega \to 0} \frac{\Delta(v,\omega,\Omega)}{2\omega^2}. \quad (17)$$

In the Gaussian $C_1$ approximation, one decouples all field averages into the two-field average $C(\Omega) = \langle E_{v-\Omega}E_{v+\Omega}^* \rangle$. Some algebra then confirms Eq. (11). We will use Eq. (17) to find the first non-Gaussian correction to this correlation function, to be referred to as $F_2(\Omega)$.

The standard recipe for calculating four-field correlators in the $C_2$ approximation has been described by Berkovits and Feng [4]. We will follow their analysis and notation, with the technical difference that we will deal with four (rather than two) different frequencies. The mechanism responsible for the non-Gaussian correction is an exchange of momenta among the four fields, as described by the four-point Hikami box $H(q_1, q_2, q_3, q_4)$ (Fig. 3). Such an event makes different wave trajectories correlated and would be excluded by Gaussian statistics. The correct expression for the vertex $H$ was given by Nieuwenhuizen and Van Rossum [33].

$$H(q_1, q_2, q_3, q_4) = -N_{H}(q_1, q_2 + q_3, q_4). \quad (18)$$

Here $N_H = \xi^4/(48\pi k^3)$. If we denote the four-field correlation by $C_2(|\Delta \nu|)$ we have, in wave number space,

$$C_2(|\Delta \nu|) = 2 \times C_1(q_1, |\Delta \nu|_1) C_2(q_2, |\Delta \nu|_2) \times H(q_1, q_2, q_3, q_4) C_3(|\Delta \nu|_3) \times C_4(q_4, |\Delta \nu|_4). \quad (19)$$

For the object in Fig. 3 one has $\{\Delta \nu_1\} = \{\omega + \Omega, \omega - \Omega, \omega, \omega\}$. The Fourier transform of this object gives the correlations in real space, and finally correlations in transmission or reflection. Following Ref. [4] for transmission through a slab with length $L$ and surface $A = L^2$, this procedure leads to

$$C_{2}^{\text{trans}}(\{\Delta \nu\}) = 2 \left( \frac{2\pi}{k} \right)^4 \left( \frac{\nu}{4\pi} \right)^4 N_H \int d^3 r \times d^3 \rho_1 \cdots d^3 \rho_4 C_2(|\Delta \nu|, \{\rho_i\}, \{z_i\}, r, |\Delta \nu|). \quad (20)$$

In this equation $r$ denotes the position of the Hikami vertex in the medium, and $\{z_i\} = \{z, L-z, L-z, L-z\}$ denote approximate entrance and exit depths of the four waves in the slab. Equation (20) can be transformed into

$$C_{2}^{\text{trans}}(|\Delta \nu|) = |T_{ab}|^2 \frac{2}{g} [I(\beta_1, \beta_2, \beta_3, \beta_4) + I(\beta_3, \beta_2, \beta_1, \beta_4)], \quad (21)$$

with $\beta_i = \sqrt{-i\Delta \nu_L^2/D + (L/L_a)^2}$ and

$$I(\beta_1, \beta_2, \beta_3, \beta_4) = \frac{\beta_1 \beta_2}{\sinh \beta_1 \cdots \sinh \beta_4} \times \int_0^1 dx \cosh \beta_1 (1-x) \cosh \beta_2 (1-x) \times \sinh \beta_3 x \sinh \beta_4 x. \quad (22)$$

We have also introduced the average one-channel transmission coefficient $|T_{ab}|^2 = (3\pi|A k^2|)^{-1}$ and the dimensionless conductance $g = (A k^2 / 3\pi) / L$. The normalized $C_2$ frequency correlation function $F_2(\Omega)$ for $W_{ab}$ in transmission can now be obtained from Eq. (17).
\[ F_2(\Omega) = \lim_{\omega \to 0} \frac{C_{\text{trans}}^2(\Omega, -\Omega, \omega, -\omega) - C_{\text{trans}}^2(\Omega + \omega, \omega - \Omega, \omega, \omega)}{\omega^2}. \]  

The limit can be carried out analytically. What remains is an integral of the kind (22) that is easily done numerically.

The special case \( \Omega = 0 \) with no absorption \( (L_a = \infty) \) can be handled analytically. The result is

\[ \langle \hat{W}_{\text{ab}}^2 \rangle = \frac{11}{5} + \frac{64}{35g} + O\left(\frac{1}{g^2}\right). \]

This can be compared to the similar expression for \( \langle \hat{P}_{\text{ab}}^2 \rangle = 2 + 4/3g \). It can be inferred that delay-time fluctuations cause the fluctuations in \( W \) to exceed those ones in \( I \), in both the \( C_1 \) and \( C_2 \) approximations. In Fig. 1 we show \( F_2(0) \) as a function of absorption. We notice that \( F_2 \) is more sensitive to absorption than \( F_1 \).

In Fig. 4 we show several frequency correlation functions for \( \hat{W}_{\text{ab}} \) (solid lines) in transmission of a slab without absorption. In contrast to the correlation function of \( \phi ' \), no singularity is encountered at \( \Omega = 0 \). Also shown (dashed) are the correlation functions of the intensity \( I_{\text{ab}} \). These correlation functions are similar, but the correlation of \( I_{\text{ab}} \phi _{\text{ab}} \) exceeds that of \( I_{\text{ab}} \) at small frequency differences, whereas it becomes smaller at large frequencies differences. The \( F_2 \) correlation function decays as \( 1/\sqrt{\Omega(\phi')} \), quite similar to the long-range correlation function of the intensity [5]. In Ref. [1] this non-Gaussian theory was shown to agree with microwave measurements.

V. CONCLUSIONS AND PROSPECTS

In conclusion, we have presented a theory for the statistics of the phase delay time of diffusing waves starting from a joint Gaussian distribution for complex fields. The sensitivity of phase to parameters other than frequency (e.g., space, time, or magnetic field) may be a useful application of this theory. We have also presented a non-Gaussian extension for the statistics of \( I d\phi dl\omega \), i.e., the intensity weighted delay time. The overall conclusion is that the delay time exhibits large mesoscopic fluctuations. Multichannel correlations, the statistics of \( N(\omega) \), and a comparison to random matrix theory are logical continuations of this work. We consider the frequency derivative of the phase, which equals the single channel delay time in the narrow band limit, to be the basic statistical dynamical variable. It is a great challenge to study the statistics of arbitrary wave packets.

ACKNOWLEDGMENTS

We thank Frédéric Faure, Roger Maynard, Ad Lagendijk, and Gabriel Cwilich for discussions. This work was supported by the Groupement de Recherches POAN and the National Science Foundation under Grant Nos. DMR 9632789 and INT9512975.

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[33] Th. M. Nieuwenhuizen and M. van Rossum, Phys. Lett. A 177, 102 (1993). It is important for our discussion that the Hikami vertex in principle also depends on the four frequencies in Fig. 3. This dependence is cancelled by realizing that the C correlators obey a diffusion equation.