

Single Parameter Scaling in One-Dimensional Localization Revisited

Lev I. Deych,¹ A. A. Lisiansky,² and B. L. Altshuler³

¹*Physics Department, Seton Hall University, South Orange, New Jersey 07052*

²*Physics Department, Queens College of CUNY, Flushing, New York 11367*

³*Physics Department, Princeton University and NEC Research Institute, Princeton, New Jersey 08540*

(Received 10 September 1999)

The variance of the Lyapunov exponent is calculated exactly in the one-dimensional Anderson model with random site energies distributed according to the Cauchy distribution. We find a new significant scaling parameter in the system, and derive an exact analytical criterion for single parameter scaling which differs from the commonly used condition of phase randomization. The results obtained are applied to the Kronig-Penney model with the potential in the form of periodically positioned δ functions with random strength.

PACS numbers: 72.15.Rn, 05.45.Ac

The hypothesis of single-parameter scaling (SPS) in the context of transport properties of disordered conductors was developed in Ref. [1]. It was suggested that there exists a single parameter, conductance g , which determines a scaling trajectory $g(L)$, where L is a size of a sample. It was soon understood [2] that scaling in the theory of localization must be interpreted in terms of the entire distribution function of the conductance. In order to take the fluctuations of the conductance into account, it is convenient to consider a parameter $\tilde{\gamma}(L) = (1/2L) \ln(1 + 1/g)$ instead of g itself [2]. The main properties of this parameter are that its limit $\gamma = \tilde{\gamma}(L \rightarrow \infty)$ is nonrandom [3] and that it has a normal limiting distribution for $L \gg \gamma^{-1}$ [4]. SPS in this situation implies that the distribution function of g is fully determined by the mean value of $\tilde{\gamma}(L)$. This mean value, which is the scaling parameter, is close to the limiting value γ , provided that $L \gg \gamma^{-1}$. The parameter γ is called the Lyapunov exponent (LE), and its inverse is the localization length, $l_{\text{loc}} = \gamma^{-1}$, of a particle's wave function. Below the random function $\tilde{\gamma}(L)$ will be referred to as the finite-size LE. The localization length of the Anderson model (AM) has been calculated by many authors (see Refs. [3,5,6]). SPS presumes that the variance, σ^2 , of $\tilde{\gamma}(L)$ is not an independent parameter but is related to LE in a universal way:

$$\sigma^2 = \gamma/L. \quad (1)$$

This relationship was first derived in Ref. [2] within the so-called random phase hypothesis, which assumes that there exists a microscopic length scale over which phases of complex transmission and reflection coefficients become completely randomized. Under similar assumptions, Eq. (1) was rederived later by many authors for a number of different models. According to the random phase model, if the phase randomization length, l_{ph} , is smaller than the localization length, l_{loc} , then Eq. (1) is valid. An analytical derivation of the distribution function of LE for the continuous Schrödinger equation [3] and numerical simulations of tight-binding AM [7] showed that the

inequality $l_{\text{ph}}\gamma \gg 1$ holds as long as disorder remains weak. Accordingly, SPS was believed to violate only at strong disorder, when l_{loc} becomes microscopic. However, recent numerical simulations for a disordered Kronig-Penney model (KPM) [8] demonstrated that, contrarily to the existing picture, even when disorder is weak and l_{loc} is large, a strong violation of SPS is still possible.

In this paper we demonstrate that the condition for the validity of SPS based upon the phase randomization concept is not accurate. We study an emergence and violation of SPS in one-dimensional systems without the assumption of phase randomization. We calculate exactly the variance of the finite size LE for AM with the Cauchy distribution of site energies (the Lloyd model) and derive Eq. (1) without any *ad hoc* hypothesis. This calculation also produces an analytic criterion for SPS in the form $\gamma^{-1} \gg l_s$. The new length l_s , which is different from the phase randomization length, is a new significant length scale. We show that even in the limit of weak disorder, when the localization length is macroscopic, the states at the tails of the spectrum never obey SPS.

One-dimensional models with off-diagonal disorder (random hopping models) represent a special case. These models demonstrate a delocalization transition in the vicinity of the zero energy state [9], which results in a violation of SPS [10] as well as in other unusual phenomena. In this paper we focus upon regular one-dimensional situations, which include models with diagonal disorder as well as random hopping models far away from the critical point, $E = 0$.

SPS also takes place in the regime of weak localization, which occurs at weak disorder in two- and three-dimensional electron systems, as well as in quasi-one-dimensional wires in the limit of a large number of conducting channels [11]. In this case, SPS manifests itself in the form of universal conductance fluctuations [12,13]. In the weak localization limit, the mean conductance coincides with $(\gamma L)^{-1}$ and remains the only significant parameter, which also determines non-Gaussian

corrections to the high moments of the distribution function [14]. SPS is violated only for very higher moments, when the far tails of the distribution function become important [14,15]. This situation is also out of the scope of the present paper, since the weak localization limit cannot be realized in one-dimensional models.

Calculation of the variance of the Lyapunov exponent for an exactly solvable model.—In this paper we discuss a quantum particle on a chain of sites, which is described by the following equation:

$$\psi_{n+1} + \psi_{n-1} - U_n \psi_n = 0, \quad (2)$$

where ψ_n represents the wave function of the system at the n th site. The meaning of U_n depends upon the interpretation of the model (2). AM corresponds to Eq. (2) with $U_n = -E + \epsilon_n$, where E is the energy of the particle, and ϵ_n is a random site energy. Another realization of Eq. (2) is a KPM with the potential given by the sum, $\sum V_n \delta(x - na)$, of periodically positioned δ functions with random strengths, V_n . In this case U_n is defined as $U_n = 2 \cos(ka) + (V_n/k) \sin(ka)$, where $k = \sqrt{E}$ is an energy variable, and a is the period of the structure [3,16]. We assume that the parameters ϵ_n or V_n are distributed with the Cauchy probability density (the Lloyd model):

$$P_C(x) = \frac{1}{\pi} \frac{\Gamma}{(x - \xi_0)(x - \xi_0^*)}, \quad (3)$$

where $\xi_0 = x_0 + i\Gamma$, and $\xi_0^* = x_0 - i\Gamma$. Parameters x_0 and Γ represent the mean value and the width of the distribution, respectively. For AM $x_0 = 0$, whereas for the KPM $x_0 = V_0$. The distribution of parameters U_n also has the form of Eq. (3), where we now denote U_0 as the center of the distribution, and Γ_U as its width. Specific expressions for U_0 and Γ_U in both AM and KPM can be obtained straightforwardly.

Below, we calculate the mean value, γ , and the variance, σ^2 , of the finite size LE, $\gamma(L)$, in the limit $L \gg \gamma^{-1}$ (further, all lengths are normalized by the lattice constant, a). Following Ref. [3] we present average LE as

$$\gamma = \langle \ln|z_n| \rangle, \quad z_n = \psi_n / \psi_{n-1},$$

and averaging $\langle \dots \rangle$ is carried out over the stationary distribution of the random variables z_n . It turns out [3] that z_n are distributed according to Eq. (3) with the complex parameter ξ_0 replaced by ξ_{st} , which satisfies the equation

$$\xi_{st} + \xi_{st}^{-1} = U_0 + i\Gamma_U. \quad (4)$$

It is convenient to parametrize ξ_{st} as

$$\xi_{st} = p \exp(i\varphi). \quad (5)$$

In this notation $\gamma = \ln|p|$. Both $p = |\xi_{st}|$ and the phase, $\varphi(E) = \arg(\xi_{st})$, depend on the energy, E .

The variance σ^2 can be expressed in terms of z_n as

$$\sigma^2 = \frac{2}{L^2} \sum_{n=0}^{N-1} \sum_{k=1}^{N-n} \langle \ln|z_n| \ln|z_{n+k}| \rangle + \frac{1}{L} \langle \ln^2|z_n| \rangle - \gamma^2. \quad (6)$$

Therefore, σ^2 can be expressed through the two point distribution, $P_2(z_n, z_k)$, of the parameters z_n . A joint distribution of multiple random variables can be expressed in terms of the product of marginal and conditional distributions. The latter probability distribution of z_k , under the condition that z_n is fixed, can be shown to satisfy the following recurrent relation ($k > n$):

$$P(z_n|z_{k+1}) = \int P(z_n|z_k) P_C(z_{k+1} + z_k^{-1}) dz_k, \quad (7)$$

where P_C is the Cauchy distribution, Eq. (3). The advantage of the Cauchy distribution is that the recurrence relation (7) can be solved exactly. The solution again has the form of the Cauchy distribution, Eq. (3):

$$P(z_n|z_k) = \frac{\Gamma_{k-n}}{\pi} \frac{1}{(z_k - \xi_{k-n})(z_k - \xi_{k-n}^*)}, \quad (8)$$

with complex parameters ξ_k obeying the equation

$$\xi_k + \xi_{k-1}^{-1} = U_0 + i\Gamma_U, \quad (9)$$

and $\Gamma_k = \text{Im}(\xi_k)$. The same Eq. (9) determines parameter ξ_{st} of the one-point distribution of quantities z_n [3]. In the latter case, however, one looks for the stationary solution of the equation, while the conditional distribution requires solution of Eq. (9) with the initial condition $\xi_0 = z_n$. With the use of the evaluated two-point probability distribution, the correlator $\langle \ln|z_n| \ln|z_{n+k}| \rangle$ can be represented as

$$\langle \ln|z_n| \ln|z_{n+k}| \rangle = \frac{\text{Im}\xi_{st}}{\pi} \int_{-\infty}^{\infty} \frac{\ln|z| \ln|\xi_k(z)|}{(z - \xi_{st})(z - \xi_{st}^*)} dz.$$

One can substitute this correlator into Eq. (6) and sum it over k without further assumptions. After calculating the average $\langle \ln^2|z_n| \rangle$, the variance σ^2 can be expressed through the parameters p and φ [Eq. (5)]:

$$\sigma^2 = \frac{1}{L} \left\{ -\gamma \ln \frac{p^4 - 2p^2 \cos(2\varphi) + 1}{(p^2 - 1)^2} + \int_{\varphi}^{\pi} dx \arctan \left[\frac{\beta}{\zeta \cos \varphi - \cos x} \right] \right\} + O(1/L^2). \quad (10)$$

Here we are interested in the limit $\gamma \ll 1$; i.e., the localization length is large $l_{loc} \gg a = 1$. In this limit the parameters $\beta(\gamma, \varphi)$ and $\zeta(\gamma, \varphi)$ are equal to $\beta \approx 2\gamma \sin \varphi$ and $\zeta \approx 1 + 2\gamma^2$, respectively. We also assume that $\varphi \leq \pi/2$; the case $\varphi \geq \pi/2$ can be handled by the replacement $\varphi \rightarrow \pi - \varphi$. Equation (10) is the main result of our calculations. It presents the asymptotically exact ($L \rightarrow \infty$) expression for the variance of LE in the Lloyd model.

Discussion.—The behavior of the integral in Eq. (10) is governed by the parameter

$$\kappa = (\gamma l_s)^{-1}, \quad (11)$$

where l_s is a new length scale in the system:

$$l_s = 1/\sin\varphi = |\xi_{st}|/\text{Im}\xi_{st}, \quad (12)$$

and ξ_{st} is given by Eq. (4). In the limit $l_s \ll \gamma$, i.e.,

$$\kappa \gg 1, \quad (13)$$

Eq. (10) reduces to

$$\sigma^2 = 2\gamma/L, \quad (14)$$

implying validity of SPS. Thus, Eq. (13) represents a true criterion for SPS.

It should be noted, however, that our result, Eq. (14), differs from Eq. (1) by the factor of 2. This difference reflects the well-known peculiarity of the Cauchy distribution—all of its moments, except for the first one, diverge. As a result, neither the approximation of the Gaussian white noise [3] nor a weak disorder expansion [7] used to derive Eq. (1) can be applied to the Lloyd model. We have calculated the variance of LE for AM with the Cauchy distribution of site energies using the random phase hypothesis, and following the approach of Ref. [7]. We found that, though a proportionality between the variance and LE persists, the numerical coefficient differs from both Eq. (1) and Eq. (14). This result implies that the phase randomization model is not valid at all for the Lloyd model. More important, however, is the fact that SPS holds even when the phase randomization hypothesis fails. This is an additional confirmation of the fact that the real criterion for SPS is given by the inequality (13) rather than by phase randomization.

Even though the localization properties of the Lloyd model are the same as those of generic models [17], one might question the generality of the new scale l_s and criterion (13) in light of the peculiarity of the Lloyd model. In order to confirm a generic nature of applicability of Eq. (13), we carried out numerical simulations of KPM with rectangular barriers of random widths. Statistics and the shape of potential in these calculations are considerably different from the Lloyd model. The results of the simulations are shown in Fig. 1 along with $\tau(\kappa) = \sigma^2 L/(2\gamma)$, obtained from our analytical Eq. (10). One can see from Fig. 1 that the crossover between different asymptotes occurs in the same region for both models. This allows us to conclude that the crossover length l_s and criterion (13) retain their significance beyond the Lloyd model.

According to Thouless [18], the phase $\varphi(E)$ is proportional to the integrated density of states $\varphi(E) = \pi G(E)$ [in the case of KPM $\varphi(E)$ must be reduced to the interval $[0, \pi]$]. For the states close to the center of the initial conductivity bands, $\varphi(E) \sim \pi/2$ and, consequently, $l_s \sim 1$. Inequality (13) reduces then to $l_{loc} \gg 1$, which essentially implies that disorder is weak. However, as soon as the energy approaches a band edge, l_s grows signifi-

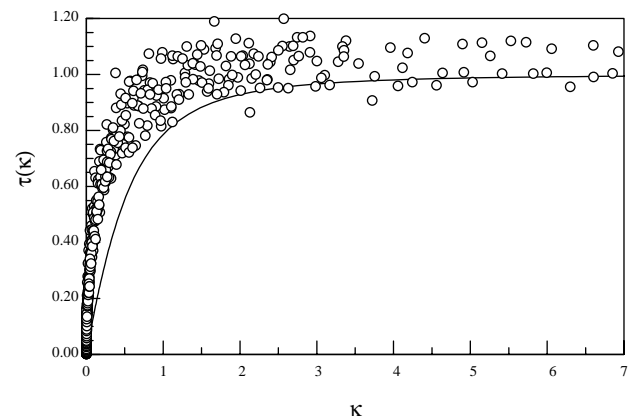


FIG. 1. The function $\tau(\kappa)$ obtained from the analytical solution of the Lloyd model Eq. (10) (solid line) and the numerical simulations of KPM with rectangular barriers of random widths (open circles). Numerical results were rescaled by the factor of 2 in order to make it clear that, though the two different models give different asymptotes of the $\tau(\kappa)$ function, the crossover length is the same for both models.

cantly, so that $\kappa \ll 1$ can coexist with weak disorder, $l_{loc} \gg 1$, when $\varphi \ll 1$ or $\pi - \varphi \ll 1$. These inequalities hold for both AM and KPM at energies outside the initial spectrum, i.e., at the tails of the density of states for AM and within former band gaps for KPM. In this case $l_s(E)$ can be expressed in terms of the number of states between E and the closest fluctuation spectrum boundary. For AM with the Cauchy distribution these boundaries are at $\pm\infty$. For KPM their role is played by energies at which $\varphi(E)/\pi$ is an integer. The states in these regions arise due to rare realizations of the disorder, and can be associated with spatially localized and well-separated structural defects. The length l_s can then be interpreted as an average distance between such defects. In view of this interpretation of l_s , the transition between two types of scaling regimes occurs when the average distance between these defects becomes comparable with the localization lengths of the respective states.

The variance σ^2 in the limit opposite to Eq. (13) can be conveniently presented in terms of the function $\tau(\kappa)$

$$\tau = \kappa \left(\frac{\pi}{2} - \kappa \right). \quad (15)$$

It is, thus, determined by the scale l_s rather than the localization length l_{loc} . This equation describes the transition from SPS to scaling with two independent parameters l_{loc} and l_s . The particular form of the function $\tau(\kappa)$ is probably model dependent. At the same time, our results suggest that κ is a universal parameter, which naturally describes the crossover between different types of statistics in the localization parameter.

The previous criterion for SPS based upon the phase randomization length [2,7] effectively restricts the strength of disorder [3,7]. The new criterion, Eq. (13), separates all the states of the system with a given strength of the

disorder into two groups, which demonstrate distinct scaling properties. The boundary between the groups, defined by the condition $\kappa = 1$, coincides with the boundaries of the initial spectrum $|U_0| = 2$, provided that $\Gamma_U \ll 1$. For the states inside the original spectra, SPS holds as long as the disorder remains weak. The length scale l_s in this case is of the order of the lattice constant a and, thus, has no scaling significance. Nevertheless, it is important to notice that this length scale differs from the phase randomization length. Indeed, it was found numerically in Ref. [7] that l_{ph} increases toward the center of the band $E = 0$ as $1/E$. Absence of the phase randomization for $E = 0$ was also obtained analytically in Ref. [19]. Both numerical, Ref. [7], and analytical, Ref. [19], calculations suggest that SPS still holds at $E = 0$, in spite of the fact that the phase does not randomize and the standard weak-disorder expansion for LE fails [5,6] at this energy. At the same time, applying criterion (13) one finds that SPS in this case violates only when disorder is so strong that the localization length becomes of a microscopic magnitude, $l_{\text{loc}} \simeq 1$ in accord with Ref. [7]. Indeed, $l_s \simeq 2/\sqrt{4 - U_0^2}$ and decreases toward the center of the band, where it assumes the minimum value $l_s = 1$. This allows us to conclude that initially conducting states always demonstrate SPS in a meaningful scaling regime $l_{\text{loc}} \gg 1$.

The situation for the states from the former band gaps is totally different: SPS violates when both lengths, l_s and l_{loc} , are macroscopic. Such a violation is significant from the scaling point of view. In the case of KPM, l_{loc} remains macroscopic throughout entire band gaps, provided that the energy is sufficiently high, $k \gg V_0$. Contrarily, $l_{\text{loc}} \sim 1$ for those states of AM that are not very far from the boundary. A strong violation of SPS in this case coincides with the system being driven out of the scaling regime.

Change in the strength of disorder affects the two scaling lengths l_{loc} and l_s differently. In the case of the gap states, l_{loc} only weakly depends upon disorder. It is approximately equal to the penetration length, which would describe tunneling through the system in the absence of disorder. Disorder related corrections to this quantity are of the order of Γ_U^2 . The parameter l_s^{-1} , on the contrary, is linearly proportional to Γ_U . Increase in disorder, therefore, causes the critical parameter κ to increase. Thus, the system can cross over between $\kappa \ll 1$ and SPS ($\kappa \gg 1$) regimes with an enhancement of the disorder. The results seem paradoxical since the restoration occurs because of increasing disorder, which must, however, remain small enough to keep the system within the scaling regime. This effect is also more important for KPM than for AM, and was observed numerically in Ref. [8] for a periodic-on-average random system.

Another interesting feature, which is specific for KPM and absent in AM, is a presence of resonance states where

$\gamma = 0$ regardless of the disorder. Though this feature is unstable with regard to a violation of the strict periodicity in the position of the δ functions, it is also present in some other models, such as random superlattices, and deserves a discussion. At the resonance, both γ and l_s^{-1} vanish. Their ratio, κ , however, remains finite and experiences a jump at the resonance point from the value of $2V_0/\Gamma \gg 1$ at the band side of the resonance to $\Gamma/2V_0 \ll 1$ at the gap side. It means that the change of statistics of LE at the resonance states from SPS behavior at the band side to two parameter scaling at the gap side occurs discontinuously.

We are indebted to A. Mirlin for a useful discussion. We also wish to thank S. Schwarz for reading and commenting on the manuscript. Work at Queens College was supported by a CUNY collaborative grant and PSC-CUNY research award. Work at Princeton University was supported by ARO under Contract No. DAAG 55-98-1-0270.

-
- [1] E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
 - [2] P.W. Anderson, D.J. Thouless, E. Abrahams, and D.S. Fisher, Phys. Rev. B **22**, 3519 (1980).
 - [3] I.M. Lifshitz, S.A. Gredeskul, and L.A. Pastur, *Introduction to the Theory of Disordered Systems* (Wiley, New York, 1988).
 - [4] P.A. Mello, J. Math. Phys. (N.Y.) **27**, 2876 (1986).
 - [5] M. Kappus and F. Wegner, Z. Phys. B **45**, 15 (1981).
 - [6] B. Derrida and E. Gardner, J. Phys. (Paris) **45**, 1283 (1984).
 - [7] A.D. Stone, D.C. Allan, and J.D. Joannopoulos, Phys. Rev. B **27**, 836 (1983).
 - [8] L.I. Deych, D. Zaslavsky, and A.A. Lisyansky, Phys. Rev. Lett. **81**, 5390 (1998).
 - [9] J.P. Bouchaud, A. Comtet, A. George, and P. Le Doussal, Ann. Phys. (N.Y.) **201**, 285 (1990); D.S. Fisher, Phys. Rev. B **50**, 3799 (1994); L. Balents and M.P.A. Fisher, Phys. Rev. B **56**, 12970 (1997); M. Steiner, M. Fabrizio, and A.O. Gogolin, Phys. Rev. B **57**, 8290 (1998).
 - [10] P.W. Brouwer, C. Mudry, and A. Furusaki, cond-mat/9904201.
 - [11] C.W.J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).
 - [12] B.L. Altshuler, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 530 (1985) [JETP Lett. **41**, 648 (1985)].
 - [13] P.A. Lee and A.D. Stone, Phys. Rev. Lett. **55**, 1622 (1985).
 - [14] B.L. Altshuler, V.E. Kravtsov, and I.M. Lerner, in *Mesoscopic Phenomena in Solids*, edited by B.L. Altshuler, P.A. Lee, and R.A. Webb (Elsevier, Amsterdam, 1991), p. 449.
 - [15] K.A. Muttalib and P. Wölfle, Phys. Rev. Lett. **83**, 3013 (1999).
 - [16] V.M. Gasparian, B.L. Altshuler, A.G. Aronov, and Z.A. Kasamanian, Phys. Lett. A **132**, 201 (1988).
 - [17] D.J. Thouless, J. Phys. C **16**, L929 (1983).
 - [18] D.J. Thouless, J. Phys. C **5**, 77 (1972).
 - [19] A.P. Dmitriev, JETP **68**, 132 (1989).